# Relativistic momentum

- Conservation of momentum involves constant center of mass with no external forces, and bookkeeping on how this changes between different inertial frames.
- Special relativity tells us to be more careful in shifting between different inertial frames.
- Let's pick our frame $S_1$ to have $v_{1,x} = 0$ and $p_{1,x} = 0$.
- Orthogonal to $\hat{x}$: use velocity transformation of $v_{2,y} = v_{1,y}/\gamma[1 - v\, v_{1,x}/c^2]$ to give

$$p_{2,y} = m_0 v_{2,y} = m_0 \frac{v_{1,y}}{\gamma}, \tag{1}$$

  giving $p_{1,y} = m v_{1,y} = \gamma p_{2,y}$ or

$$p_{1,y} = \gamma m_0 v_{2,y}. \tag{2}$$

  Inertial mass $m_0$ of particle in frame $S_2$ looks to us in $S_1$ as if it has increased by factor $\gamma$.
- If we brought particle to rest in our frame we would measure the same "rest mass" $m_0$.

# Relativistic forces: $\vec{F} \perp \vec{v}$

- Force is defined as the change in momentum per time:

$$\vec{F} \equiv \frac{d(m_0\gamma\vec{v})}{dt} = m_0\gamma\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt} + \gamma\vec{v}\frac{dm}{dt} \tag{3}$$

  Rest mass is the rest mass, so $dm_0/dt = 0$.
- Consider case when force is always perpendicular to velocity (for example, charged particle in a magnetic field). Now $\vec{F} \perp \vec{v}$. No velocity and therefore no motion along $\vec{F}$ direction so no work!
- Direction of $\vec{v}$ changes but magnitude does not, so $d\gamma/dt = 0$. We are therefore left with

$$\vec{F} = m_0\gamma\frac{d\vec{v}}{dt} = m_0\gamma\vec{a} \qquad (\text{for } \vec{F} \perp \vec{v}) \tag{4}$$

- Centripetal force:

$$\vec{F} = m_0\gamma\vec{a} = -m_0\gamma\frac{v^2}{r} \tag{5}$$

  giving $qvB = m_0\gamma v^2/r$ or $r = m_0\gamma v/qB$. Verified experimentally in 1909.

## $\vec{F} \parallel \vec{v}$

- Now particle speed and thus $\gamma$ will *not* be constant. Return to Eq. 3 with $dm_0/dt = 0$:

$$\vec{F} \equiv \frac{d(m_0\gamma\vec{v})}{dt} = m_0\gamma\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt}$$

  Again, $d\vec{v}/dt = \vec{a}$. Calculate $d\gamma/dt$:

$$\frac{d}{dt}(1-v^2/c^2)^{-1/2} = -\frac{1}{2}(1-v^2/c^2)^{-3/2}(-2)\frac{\vec{v}}{c^2}\frac{d\vec{v}}{dt} = \gamma^3\frac{\vec{v}}{c^2}\vec{a} \quad (6)$$

  because it's only when we have the square of velocity that we lose information on its direction.

- Returning to Eq. 3 we now have

$$\begin{aligned}
\vec{F} &= m_0\gamma\vec{a} + m_0\vec{v}\,\gamma^3\frac{\vec{v}}{c^2}\vec{a} = m_0\gamma\vec{a}\left(1+\gamma^2\frac{v^2}{c^2}\right) \quad (7)\\
&= m_0\gamma\vec{a}\left(1+\frac{v^2}{c^2-v^2}\right) = m_0\gamma\vec{a}\left(\frac{c^2-v^2}{c^2-v^2}+\frac{v^2}{c^2-v^2}\right)\\
&= m_0\gamma\vec{a}\,\frac{c^2}{c^2-v^2} = m_0\gamma\vec{a}\,\frac{1}{1-v^2/c^2} = m_0\gamma^3\vec{a}.
\end{aligned}$$

# Kinetic energy

- We now know force along direction of frame shift $\vec{F} \parallel \vec{v}$. Calculate kinetic energy from work done to move particle:

$$E_k = \int_0^{x'} F\,dx = \int_0^{x'} m_0\gamma^3 a\,dx. \tag{8}$$

- Now

$$a\,dx = \frac{dv}{dt}\,dx = \frac{dx}{dt}\,dv = v\,dv, \tag{9}$$

so we can write the kinetic energy as

$$E_k = \int_0^{v'} \gamma^3 m_0 v\,dv = m_0 \int_0^{v'} \frac{v}{(1 - v^2/c^2)^{3/2}}\,dv. \tag{10}$$

# Kinetic energy II

- Again we had from Eq. 10

$$E_k = m_0 \int_0^{v'} \frac{v}{(1 - v^2/c^2)^{3/2}} \, dv.$$

Define $A \equiv c^2(1 - v^2/c^2)^{-1/2}$ so that

$$dA = c^2(-1/2)(1 - v^2/c^2)^{-3/2}(-2v/c^2)\, dv = \frac{v}{(1 - v^2/c^2)^{3/2}} \, dv. \tag{11}$$

- Therefore, we can recognize Eq. 10 as $\int dA = A$ and obtain

$$\begin{aligned} E_k &= \frac{m_0 c^2}{(1 - v^2/c^2)^{1/2}} \Big|_0^{v'} \qquad (12) \\ &= m_0 c^2 \left( \frac{1}{(1 - v'^2/c^2)^{1/2}} - 1 \right) \\ &= (\gamma - 1) m_0 c^2. \end{aligned}$$

# Correspondence principle and kinetic energy

- In classical limit $v \ll c$, we found found $\gamma \simeq 1 + \frac{1}{2}\frac{v^2}{c^2}$.
- Therefore the classical limit of relativistic kinetic energy is

$$E_k \simeq \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) m_0 c^2 = \frac{1}{2} m_0 v^2 \tag{13}$$

  as expected (**correspondence principle**).
- In the highly relativistic limit of $\gamma \gg 1$, we instead obtain

$$E_k \simeq \gamma m_0 c^2. \tag{14}$$

# Particles in motion

- $E_k = (\gamma - 1)m_0c^2$ suggests a new interpretation of the total energy of a particle in motion.
- Assume that a particle at rest has some energy $E_0$ associated with it:

$$E = E_0 + E_k. \tag{15}$$

- For $\gamma \gg 1$, we found $E \simeq E_k \simeq \gamma m_0 c^2$. Therefore, we make the association $E_0 = m_0c^2$ which allows us to write the total energy as

$$E = E_0 + E_k = m_0c^2 + (\gamma - 1)m_0c^2 \tag{16}$$

or the sum of rest and kinetic energy. We can also write $E = \gamma m_0 c^2$ for the total energy.

# Rest energy and electron-Volts

- $E_0 = m_0 c^2$ is probably the best-known result of modern physics. "Weigh" particles in natural units for atomic and nuclear physics calculations.
- The electron-Volt, or eV: energy gained by an electron as it experiences an electrostatic potential change of one volt. Work is $W = qV$, giving

$$1 \text{ eV} = 1.6 \times 10^{-19} \frac{\text{Coulomb}}{e^- \text{ charge}} \cdot 1 \text{ Volt} = 1.6 \times 10^{-19} \text{ Joule.} \quad (17)$$

- Proton mass in eV:

$$\begin{aligned} E_0 &= m_0 c^2 = 1.67 \times 10^{-27} \text{ kg} \cdot (3 \times 10^8 \text{ m/sec})^2 \quad (18) \\ &= 1.5 \times 10^{-10} \text{ Joules} \cdot \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ Joules}} \\ &= 939 \times 10^6 \text{ eV.} \end{aligned}$$

The mass $m_0 = E_0/c^2$ can then be written as $m_0 = 939 \text{ MeV}/c^2$. Sloppy version: "the proton mass is 939 MeV" or "the electron mass is 511 keV."

# Chemical reactions

- Some chemical bond energies: H-C bond 80.9 kcal/mol, C-N bond 184 kcal/mol, C-O 257 kcal/mol.
- Convert H-C to kJ/mol:

$$80.9 \, \frac{\text{kcal}}{\text{mol}} \cdot 4.184 \, \frac{\text{kJ}}{\text{kcal}} = 338 \text{ kJ/mol}.$$

- Convert to eV/atom:

$$338 \, \frac{\text{kJ}}{\text{mol}} \cdot \frac{10^3 \text{ J}}{\text{kJ}} \cdot \frac{1 \text{ eV}}{1.602 \times 10^{-19} \text{ J}} \cdot \frac{1 \text{ mol}}{6.02 \times 10^{23}} = 3.50 \text{ eV}/\text{bond}$$

- Equating to $m_0 c^2$ gives a fractional mass change for electron of

$$\frac{3.50 \text{ eV}}{511 \times 10^3 \text{ eV}} = 7 \times 10^{-6}$$

- Difficult to detect any mass change due to chemical bonding! **Correspondence principle** in action again.

# Relativistic conservation of momentum

- Classically, kinetic energy is $p^2/2m$. Consider $p^2$ in relativity:

$$(pc)^2 = (\gamma m_0 vc)^2 = (\gamma\beta m_0 c^2)^2. \tag{19}$$

- If we then use $E_0 = m_0c^2$ and $\beta^2 = 1 - \frac{1}{\gamma^2}$, we obtain

$$p^2c^2 = \gamma^2(1 - \frac{1}{\gamma^2})E_0^2 = (\gamma^2 - 1)E_0^2 = \gamma^2 E_0^2 - E_0^2. \tag{20}$$

  However, since we found before that the total energy is $E = \gamma E_0$, we have $p^2c^2 = E^2 - E_0^2$ or

$$E^2 = E_0^2 + p^2c^2. \tag{21}$$

- Therefore if $E_k \gg E_0$ we have $E_{k,\text{relativistic}} \simeq pc$.

# Mass of the photon

- Mass of something that is allowed to travel at the speed of light? From Eq. 16 we get $E = \gamma E_0$ or

$$E_0 = \frac{E}{\gamma} = E\sqrt{1 - v^2/c^2}. \tag{22}$$

  With $v = c$, we get $E_0 = E\sqrt{1-1} = 0$, so that the rest mass $m_0$ must also be zero.
- Because light must travel with a velocity of $c$, we therefore conclude that photons have a rest mass of zero.

# Energy–momentum invariant

Rearrange Eq. 21 to give

$$\begin{aligned}\left(\frac{E_0}{c}\right)^2 &= \left(\frac{E}{c}\right)^2 - p^2 \\ &= \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2. \end{aligned} \tag{23}$$

$(E_0/c)^2 = (m_0 c)^2$ has the same value when measured in any inertial frame; therefore so does the quantity on the right hand side of Eq. 23. Thus

$$\left(\frac{E_1}{c}\right)^2 - p_{x,1}^2 - p_{y,1}^2 - p_{z,1}^2 = \left(\frac{E_2}{c}\right)^2 - p_{x,2}^2 - p_{y,2}^2 - p_{z,2}^2 \tag{24}$$

between frames $S_1$ and $S_2$. This is really the equivalent of our statement in a previous lecture

$$(ct_1)^2 - x_1^2 - y_1^2 - z_1^2 = (ct_2)^2 - x_2^2 - y_2^2 - z_2^2 \tag{25}$$

which served as the basis for our derivation of the Lorentz transformations!

# Momentum transforms

We went from $(ct_1)^2 - x_1^2 - y_1^2 - z_1^2 = (ct_2)^2 - x_2^2 - y_2^2 - z_2^2$ to find transformations for position and time:

$$\begin{aligned} x_2 &= \gamma\left(x_1 - vt_1\right) \quad \text{and} \quad y_2 = y_1 \quad \text{and} \quad z_2 = z_1 \\ t_2 &= \gamma\left(t_1 - \frac{\beta}{c}x_1\right) \end{aligned}$$

From Eq. 24 of $(E_1/c)^2 - p_{x,1}^2 - p_{y,1}^2 - p_{z,1}^2 = (E_2/c)^2 - p_{x,2}^2 - p_{y,2}^2 - p_{z,2}^2$ we can find an equivalent Lorentz transform for momentum and energy:

$$\begin{aligned} p_{x,2} &= \gamma\left(p_{x,1} - v\left(\frac{E_1}{c^2}\right)\right) \quad \text{and} \quad p_{y,2} = p_{y,1} \text{and} \quad p_{z,2} = p_{z,1} \\ E_2 &= \gamma(E_1 - vp_{x,1}). \end{aligned} \tag{26}$$

This is somewhat startling, for it tells us that we need to worry about the Lorentz transformation in considering conservation of energy!