

Einstein's 3rd paper: motivation

Einstein

Special relativity

Lorentz contraction

Time dilation

Length contraction

Velocities

First page of Einstein's 1905 paper "On the Electrodynamics of Moving Bodies" (translation from Griffiths, *Introduction to Electrodynamics*, Prentice-Hall, 1981, p. 392):

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. . .

Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relative to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest.

Einstein's postulates

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Special relativity: two frames with velocity difference (general relativity deals with large acceleration differences). Postulates:

- 1 The laws of physics are the same in all inertial reference frames.
- 2 The speed of light in free space has the same value $c = 1/\sqrt{\mu_0\epsilon_0}$ in all inertial reference frames.

Postulate 2 explains Michelson-Morley. But what about the speed of light coming out from the headlights of a spaceship?

Light pulse from a point

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At $t_1 = t_2 = 0$, a spherical wave light pulse is emitted from a common source point. Frame S_2 is moving at a speed v in the \hat{x} direction relative to frame S_1 . Positions of pulse at a later time:

$$x_1^2 + y_1^2 + z_1^2 - c^2 t_1^2 = 0 \quad (1)$$

$$x_2^2 + y_2^2 + z_2^2 - c^2 t_2^2 = 0. \quad (2)$$

Applying Galilean relativity

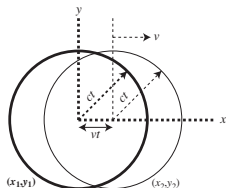
$$x_2 = x_1 - vt_1$$

$$y_1 = y_2$$

$$z_1 = z_2$$

$$t_1 = t_2 \quad (3)$$

gives inconsistencies! (Homework)



Observers in both frames see light pulse as a sphere expanding from an origin at the same velocity

How to fix it?

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Motion is in \hat{x} , so assume $y_1 = y_2$ and $z_1 = z_2$ as before. Simplest fix is a linear correction factor γ :

$$x_2 = \gamma(x_1 - vt_1) \quad (4)$$

Rewrite Eq. 2 [$x_2^2 + y_2^2 + z_2^2 - c^2t_2^2 = 0$] using the above. Equate it [= 0] with Eq. 1 [$x_1^2 + y_1^2 + z_1^2 - c^2t_1^2 = 0$] to get:

$$x_1^2 - c^2t_1^2 = 0 = \gamma^2(x_1 - vt_1)^2 - c^2t_2^2 = \gamma^2(x_1^2 - 2x_1vt_1 + v^2t_1^2) - c^2t_2^2 \quad (5)$$

Gives

$$(1 - \gamma^2)x_1^2 + 2\gamma^2vx_1t_1 - \gamma^2v^2t_1^2 + c^2(t_2^2 - t_1^2) = 0. \quad (6)$$

It is inescapable that t_2 will depend on both t_1 and x_1 .

Finding the fix

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Because t_2 must depend on both t_1 and x_1 , assume a solution for t_2 of the form

$$t_2 = At_1 - Bvx_1. \quad (7)$$

Substituting this into Eq. 6 and collecting terms, we obtain

$$(1 - \gamma^2 + c^2v^2B^2)x_1^2 + 2(\gamma^2v - c^2vAB)x_1t_1 + (c^2A^2 - c^2 - \gamma^2v^2)t_1^2 = 0. \quad (8)$$

Quadratic equation, which would have only two solutions of x_1 . But we need something that works for all x_1 ! Must make coefficients be zero:

$$(1 - \gamma^2 + c^2v^2B^2) = 0 \quad \text{or} \quad c^2B^2 = \frac{\gamma^2 - 1}{v^2} \quad (9)$$

$$(\gamma^2v - c^2vAB) = 0 \quad \text{or} \quad \gamma^4 = c^2A^2 c^2B^2 \quad (10)$$

$$(c^2A^2 - c^2 - \gamma^2v^2) = 0 \quad \text{or} \quad c^2A^2 = \gamma^2v^2 + c^2. \quad (11)$$

Simplifying

Again, we have $c^2 B^2 = (\gamma^2 - 1)/v^2$ from Eq. 9, and $c^2 A^2 = \gamma^2 v^2 + c^2$ from Eq. 11, and $c^2 A^2 c^2 B^2 = \gamma^4$ from Eq. 10. We can then substitute Eqs. 9 and 11 into Eq. 10 to get (using $\beta \equiv v/c$)

$$\begin{aligned}\gamma^4 &= c^2 A^2 c^2 B^2 = (\gamma^2 v^2 + c^2) \frac{(\gamma^2 - 1)}{v^2} \\ &= \left(\gamma^2 + \frac{1}{\beta^2}\right)(\gamma^2 - 1) = \gamma^4 + \frac{\gamma^2}{\beta^2} - \gamma^2 - \frac{1}{\beta^2} \\ \gamma^2 \left(\frac{1}{\beta^2} - 1\right) &= \frac{1}{\beta^2} \\ \gamma^2 &= \frac{1}{\beta^2} \frac{1}{1/\beta^2 - 1} = \frac{1}{1 - \beta^2}\end{aligned}$$

which finally gives

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad (12)$$

where $\beta \equiv v/c$.

Simplifying II

Let's turn back to Eq. 9, and use $\beta \equiv v/c$:

$$\begin{aligned}c^2 B^2 &= \frac{\gamma^2 - 1}{v^2} \\c^4 \beta^2 B^2 &= \gamma^2 - 1 = \frac{1}{1 - \beta^2} - 1 = \frac{1}{1 - \beta^2} - \frac{1 - \beta^2}{1 - \beta^2} = \frac{\beta^2}{1 - \beta^2} = \beta^2 \gamma^2 \\c^4 B^2 &= \gamma^2 \\B &= \frac{\gamma}{c^2}\end{aligned}\tag{13}$$

Let's also turn back to Eq. 11 of $c^2 A^2 = \gamma^2 v^2 + c^2$:

$$A^2 = \gamma^2 \beta^2 + 1 = \frac{\beta^2}{1 - \beta^2} + \frac{1 - \beta^2}{1 - \beta^2} = \frac{1}{1 - \beta^2} = \gamma^2\tag{14}$$

which gives $A = \gamma$ so that we can write our velocity transformation from Eq. 7 as

$$t_2 = A t_1 - B v x_1 = \gamma \left(t_1 - \frac{\beta}{c} x_1 \right).\tag{15}$$

Lorentz transformations

We have shown that the net transformation between coordinate systems is

$$x_2 = \gamma (x_1 - vt_1) \quad (16)$$

$$y_2 = y_1 \quad (17)$$

$$z_2 = z_1 \quad (18)$$

$$t_2 = \gamma \left(t_1 - \frac{\beta}{c} x_1 \right) \quad (19)$$

with γ found from Eq. 12 to be

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}},$$

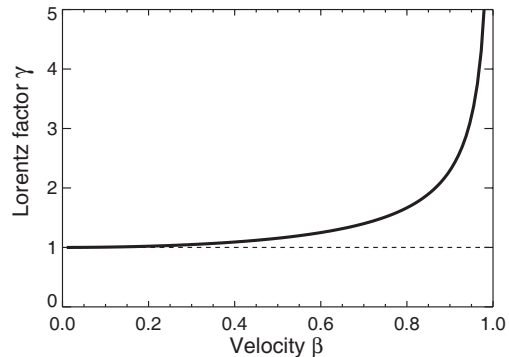
This transformation was first worked out by H. Lorentz in 1904, but in the assumption of a moving æther; as a result, Einstein gets the credit for getting the physics right. Important mathematical contributions later added by Poincaré and Minkowski.

Lorentz factor γ

In the limit of $v \ll c$ or $\beta \ll 1$, we have

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = (1 - \beta^2)^{-1/2} \simeq 1 + \frac{1}{2}\beta^2. \quad (20)$$

In the limit $\beta \rightarrow 0$, we have $\gamma \rightarrow 1$ and Eqs. 16 and 19 both revert to the Galilean results. This illustrates the all-important **correspondence principle** with classical physics.



Inverse Lorentz transformations

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Special relativity

Lorentz contraction

Time dilation

Length contraction

Velocities

Since observer at rest in S' or S_2 sees S or S_1 moving at $-v$, inverse transformations are

$$x_1 = \gamma (x_2 + vt_2) \quad (21)$$

$$y_1 = y_2 \quad (22)$$

$$z_1 = z_2 \quad (23)$$

$$t_1 = \gamma \left(t_2 + \frac{\beta}{c} x_2 \right) \quad (24)$$

Relativistic time dilation

Einstein

Special relativity

Lorentz contraction

Time dilation

Length contraction

Velocities

- Fastest way to sense someone's clock ticks: speed of light.
- We are at rest in frame 1; clock is moving at v in frame 2.
- Light pulses emitted in clock's frame at t_2 and then t'_2 , so $T_2 = t'_2 - t_2$.
- Light pulses are emitted from same point on clock so $x_2 = x'_2$.
- To us (using Eq. 19):

$$t_1 = \gamma(t_2 + \frac{\beta}{c}x_2) \quad \text{and} \quad t'_1 = \gamma(t'_2 + \frac{\beta}{c}x'_2)$$

- Our time interval T_1 :

$$T_1 = t'_1 - t_1 = \gamma\left((t'_2 - t_2) + \frac{\beta}{c}(x'_2 - x_2)\right) = \gamma T_2 \quad (25)$$

We see time dilated by γ relative to the moving frame. This is often written as $t' = \gamma t_0$.

The twin paradox

Einstein

Special relativity

Lorentz contraction

Time dilation

Length contraction

Velocities

- At age 20, Sluggo stays home, while his twin Speedo travels to the Planet of the Apes, 10 light years from Earth, at $v = 0.5c$. Speedo then returns. Ignore crushing accelerations. . .
- To Sluggo, Speedo's journey took $t = x/v = 20$ years there, and 20 years back, or 40 years total.
- Sluggo's clock is dilated by γ relative to Speedo's; since $\gamma = 1/\sqrt{1 - (0.5c/c)^2} = 1.15$, Speedo's clock ran for $40/1.15 = 34.6$ years.
- Upon Speedo's return, Sluggo is $20 + 40 = 60$ years old, while Speedo is $20 + 34.6 = 54.6$ years old!

The twin paradox: experimental verification

Einstein

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Velocities

Abstract, right? It's been tested! Atomic clocks have been flown around the world on jets. Time discrepancies in nanoseconds:

Eastward: measured -59 ± 10 nsec, predicted -40 ± 23 nsec

Westward: measured 273 ± 7 nsec, predicted 275 ± 21 nsec

See Hafele and Keating, *Science* **177**, 166 and 168 (1972). Note that the calculation involves both special relativity (plane going at a velocity relative to observer on earth's surface) and general relativity (acceleration from plane ascending and descending, and also from weaker gravitational attraction at altitude).

Time travel?

Einstein

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Velocities

*There was a young lady named Bright,
Whose speed was far faster than light;
She set out one day
In a relative way,
And returned on the previous night*

Attributed to Arthur Buller in P. Davies, “Wormholes and time machines,” *Sky & Telescope* **83**, 20 (January 1992).

Of course, as a physics teacher I tell my students that faster-than-light travel is impossible, but that’s just to crush their spirits.

[LaNelle Ohlhausen](#) in the 1997-03 issue of the *Annals of Improbable Research*.

Calvin and Hobbes

Einstein

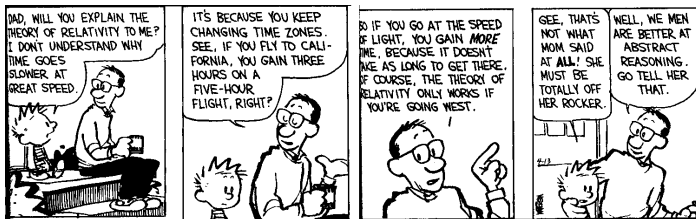
Special relativity

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calvin and hobbes



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Thanks to Sam Watterson...

Length contraction

Einstein

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Length contraction

Velocities

- We are at rest in frame 1; measuring rod with ends separated by $L_2 = x'_2 - x_2$ is moving at v in frame 2.
- We take a photograph of the rod at one instant of time, so for us $t_1 = t'_1$.
- We measure a distance $L_1 = x'_1 - x_1$.
- Using Eq. 16, we can now relate the two lengths:

$$x_2 = \gamma(x_1 - vt_1) \quad \text{and} \quad x'_2 = \gamma(x'_1 - vt'_1)$$

- Relationship between perceived lengths is thus

$$L_2 = x'_2 - x_2 = \gamma\left((x'_1 - x_1) - v(t'_1 - t_1)\right) = \gamma L_1 \quad (26)$$

We see length contracted by $L_1 = (1/\gamma)L_2$. This is often written as $l' = l_0/\gamma$.

Relativistic velocity

Einstein

Special relativity

Lorentz contraction

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Length contraction

Velocities

Take derivatives of Eqs. 16 through 19 to give

$$\begin{aligned} dx_2 &= \gamma(dx_1 - v dt_1) & \text{and} & & dy_2 &= dy_1 & \text{and} & & dz_2 &= dz_1 \\ dt_2 &= \gamma\left(dt_1 - \frac{\beta}{c} dx_1\right). \end{aligned}$$

The velocity in frame 2 is the change in position dx_2 divided by the change in time dt_2 . Divide numerator and denominator by dt_1 :

$$\begin{aligned} v_{2,x} &= \frac{dx_2}{dt_2} = \frac{\gamma(dx_1 - v dt_1)}{\gamma(dt_1 - (v/c^2)dx_1)} = \frac{(dx_1/dt_1) - v(dt_1/dt_1)}{(dt_1/dt_1) - (v/c^2)(dx_1/dt_1)} \quad (27) \\ &= \frac{v_{1,x} - v}{1 - \frac{v v_{1,x}}{c^2}} \end{aligned}$$

Relativistic velocity II

Einstein

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Velocities

Calculate $v_{2,y}$ and $v_{2,z}$ in the same manner:

$$v_{2,y} = \frac{dy_2}{dt_2} = \frac{dy_1}{\gamma [dt_1 - (v/c^2)dx_1]} = \frac{(dy_1/dt_1)}{\gamma [(dt_1/dt_1) - (v/c^2)(dx_1/dt_1)]} \quad (28)$$

$$= \frac{v_{1,y}}{\gamma \left[1 - \frac{v v_{1,x}}{c^2}\right]}$$

$$v_{2,z} = \frac{v_{1,z}}{\gamma \left[1 - \frac{v v_{1,x}}{c^2}\right]}. \quad (29)$$

Relativistic velocity: inverse equations

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Velocities

The inverses of the relativistic velocity expressions are

$$v_{1,x} = \frac{v_{2,x} + v}{1 + \frac{vv_{2,x}}{c^2}} \quad (30)$$

$$v_{1,y} = \frac{v_{2,y}}{\gamma \left[1 + \frac{vv_{2,x}}{c^2} \right]} \quad (31)$$

$$v_{1,z} = \frac{v_{2,z}}{\gamma \left[1 + \frac{vv_{2,x}}{c^2} \right]}. \quad (32)$$