PHY 251 Spring 2008: homework problem set 9, due Thursday, April 16.

Serway problem 10.2

Answer: From Serway Eq. 10.8, we know that the Maxwell-Boltzmann velocity distribution is

$$n(v)\,dv = \frac{4\pi N}{V}\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2\ \exp[-\frac{mv^2}{2k_B T}]\,dv = B\,v^2 e^{-Av^2}\,dv$$

with $A \equiv m/(2k_B T)$. The most probable speed $v_{\rm mp}$ is at the top of the curve, so we take the derivative and set to zero:

$$\begin{aligned}\frac{d}{dv}B\,v^2 e^{-Av^2}\,dv &= B\left(2ve^{-Av^2} + (-2Av)v^2 e^{-Av^2}\right) = 2vBe^{-Av^2}(1 - Av^2) = 0\\ 1 &= Av^2\\ v_{\rm mp} &= \sqrt{\frac{1}{A}} = \sqrt{\frac{2k_B T}{m}}\end{aligned}$$

Serway problem 10.3

Answer: Let the cylinder have a diameter of d. The travel time for a gas molecule with speed v to transit the cylinder is $t = d/v$. During that time t, the glass plate will have traveled over an arc length s of

$$s = r\theta = \frac{d}{2}\omega t = \frac{d}{2}\omega\frac{d}{v} = d^2\frac{\omega}{2v}.$$

We need to find the angular velocity:

$$\frac{6250\cdot 2\pi}{1\text{ min}}\ \frac{1\text{ min}}{60\text{ sec}} = 654.5\text{ rad/sec}.$$

Now we need to consider the different molecular velocities. We're working with bismuth molecules, which have a mass of

$$2\cdot(208.9\text{ amu})\cdot(1.661\times 10^{-27}\text{ kg/amu}) = 6.94\times 10^{-25}\text{ kg}.$$

We found the various velocities on slide 10 of lecture 15 to be

$$\bar{v} = \sqrt{\frac{8}{\pi}\frac{k_B T}{m}} \qquad v_{\rm mp} = \sqrt{2\frac{k_B T}{m}} \qquad v_{\rm rms} = \sqrt{3\frac{k_B T}{m}}$$

or

$$[\bar{v}, v_{\rm mp}, v_{\rm rms}] = \sqrt{\frac{k_B T}{m}\,[\frac{8}{\pi}, 2, 3]}$$

so we can find corresponding arc distances s of (using all mks units)

$$\begin{aligned}[\bar{s}, s_{\rm mp}, s_{\rm rms}] &= d^2\frac{\omega}{2v} = d^2\frac{\omega}{2}\sqrt{\frac{m}{k_B T}\,[\frac{\pi}{8}, \frac{1}{2}, \frac{1}{3}]}\\ &= (0.10)^2\frac{654.5}{2}\sqrt{\frac{6.94\times 10^{-25}}{1.381\times 10^{-23}\cdot 850}[\frac{\pi}{8}, \frac{1}{2}, \frac{1}{3}]} = [0.0158, 0.0.178, 0.0145]\end{aligned}$$

or 1.58, 1.78, and 1.45 cm respectively for $\bar{s}$, s_{mp}, and s_{rms}. If we were to use light atoms like O_2 or N_2, we would have much less mass and thus a less noticeable distance on the drum with a particular angular velocity.

Serway problem 10.6

Answer: The ratio of the number of particles in the first excited state n_2 relative to the number in the ground state n_1 is

$$\begin{aligned} x &\equiv \frac{n_2}{n_1} = \frac{g_2}{g_1}\frac{f_2}{f_1} = \frac{1}{1}\frac{\exp[-E_2/(k_B T)]}{\exp[-E_1/(k_B T)]} \\ &= \exp[(E_1 - E_2)/(k_B T)] = \exp[\big(E_1 - (E_1 + 4.86)\big)/(k_B T)] \\ &= \exp[-(4.86\text{ eV})/(8.617 \times 10^{-5} \cdot 1600)] = 4.9 \times 10^{-16} \end{aligned}$$

If there are $n_1 + n_2 = N = 10^{20}$ atoms, we can write $n_2 = AN$ and $n_1 = (1 - A)N$ and thus find

$$\begin{aligned} x &\equiv \frac{n_2}{n_1} = \frac{AN}{(1-A)N} = \frac{A}{1-A} \\ (1-A)x &= A \\ A(1+x) &= x \\ A &= \frac{x}{1+x} \simeq x \qquad \text{for} \qquad x \ll 1 \\ n_2 &= AN = (4.9 \times 10^{-16}) \cdot 10^{20} = 4.9 \times 10^4 \text{ atoms} \\ n_1 &= (1-A)N = (1 - \frac{x}{1+x})N = (\frac{1+x}{1+x} - \frac{x}{1+x})N = \frac{1}{1+x}N \simeq (1-x)N \\ &= 10^{20} - 4.9 \times 10^4 \text{ atoms} \end{aligned}$$

The power radiated is the number of de-excitations per second (n_2/τ) times the photon energy, or

$$P = \frac{n_2}{\tau} h\nu = \frac{4.9 \times 10^4}{1 \times 10^{-7}\text{ sec}} \cdot (4.86\text{ eV}) \cdot (1.602 \times 10^{-19}\text{ J/eV}) = 3.8 \times 10^{-7}\text{ Watts}$$

Serway problem 10.9

Answer: The mass of an iron atom is

$$(55.8\text{ amu}) \cdot (1.661 \times 10^{-27}\text{ kg/amu}) = 9.27 \times 10^{-26}\text{ kg}.$$

so we have

$$\bar{v} = \sqrt{\frac{8k_B T}{\pi m}} = \sqrt{\frac{8 \cdot 1.381 \times 10^{-23} \cdot 6000}{\pi \cdot 9.27 \times 10^{-26}}} = 1510\text{ m/s}$$

which is beyond the speed of sound at standard temperature and pressure (300 m/s) but nowhere near relativistic. We can therefore use the low β expansion of the Doppler shift for light waves:

$$\begin{aligned} f &= f_0 \frac{\sqrt{1 \pm \beta}}{\sqrt{1 \mp \beta}} = f_0(1 \pm \beta)^{1/2}(1 \mp \beta)^{-1/2} \\ &\simeq f_0(1 \pm \frac{1}{2}\beta)(1 \pm \frac{1}{2}\beta) \simeq f_0(1 \pm \beta) \\ \Delta f &= f - f_0 = f_0(1 \pm \beta) - f_0 = \pm \beta f_0 \end{aligned}$$

The fractional Doppler shift is then

$$\frac{\Delta f}{f_0} = \pm \beta = \pm \frac{\bar{v}}{c} = \pm \frac{1510}{2.99 \times 10^8} = \pm 5.05 \times 10^{-6}$$

Serway problem 10.15

Answer: From

$$E_F = \frac{h^2}{8m} \left(\frac{3}{\pi} \frac{N}{V} \right)^{2/3}$$

we can find

$$\begin{aligned} \frac{N}{V} &= \frac{\pi}{3} \left(\frac{8mE_F}{h^2} \right)^{3/2} = \frac{\pi}{3} \left(\frac{8mc^2 E_F}{(hc)^2} \right)^{3/2} \\ &= \frac{\pi}{3} \left(\frac{8 \cdot 511 \times 10^3 \cdot 11.63 \, \text{eV}^2}{(1240 \text{ eV} \cdot \text{nm})^2} \right)^{3/2} = 180 \, \frac{\text{electrons}}{\text{nm}^3} \, (\frac{10^7 \text{ nm}}{\text{cm}})^3 = 1.8 \times 10^{23} \, \frac{\text{electrons}}{\text{cm}^3} \end{aligned}$$

The number of atoms per unit volume is

$$\frac{\rho \cdot N_A}{A} = \frac{(2.70 \text{ g/cm}^3) \cdot (6.02 \times 10^{23} \text{ atoms/mol})}{27.0 \text{ g/mol}} = 6.02 \times 10^{22} \text{ atoms/cm}^3$$

so the number of valence electrons per aluminum atom is $(18. \times 10^{22})/(6.0 \times 10^{22}) = 3.0$.

Serway problem 10.21

Answer: For sodium, the atom number density is

$$\frac{N}{V} = \frac{\rho \, N_A}{A} = \frac{(0.971 \text{ g/cm}^3) \cdot (6.02 \times 10^{23} \text{ atoms/mol})}{23.0 \text{ g/mol}} = 2.54 \times 10^{22} \text{ atoms/cm}^3$$

and since it is monovalent rather than di- or trivalent there is one valence electron per atom. The Fermi energy is then

$$\begin{aligned} E_F &= \frac{h^2}{8m} \left(\frac{3}{\pi} \frac{N}{V} \right)^{2/3} = \frac{(hc)^2}{8mc^2} \left(\frac{3}{\pi} \frac{N}{V} \right)^{2/3} \\ &= \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8 \cdot (511 \times 10^3 \text{ eV})} \left(\frac{10^{-7} \text{ cm}}{1 \text{ nm}} \right)^2 \left(\frac{3}{\pi} \, (2.54 \times 10^{22} \, \frac{1}{\text{cm}^3} \right)^{2/3} = 3.15 \text{ eV} \end{aligned}$$

The Fermi speed is found from $(1/2)mv^2 = E_F$ which gives a Fermi speed of

$$v_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2 \cdot (3.15 \text{ eV})}{511 \times 10^3 \text{ eV/c}^2}} = 0.00351c = 1.05 \times 10^6 \text{ m/s.}$$

Serway problem 12.16

Answer: The longest wavelength is 1.85 μm which corresponds to a minimum photon energy of

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{1850 \text{ nm}} = 0.67 \text{ eV}.$$

This is the energy gap between bands.

Serway problem 12.18

Answer: The idea here is to replace Z with Z/κ and use $Z = 1$ because of screening of the nuclear charge by all the bound electrons. Thus for $n = 1$ we have

$$E = (-13.6 \text{ eV})Z^2 \; \Rightarrow \; (-13.6 \text{ eV})(\frac{1}{\kappa})^2$$

so for Si we have $13.6/12^2$=0.094 eV and for Ge we have $13.6/16^2$=0.053 eV as the effective binding energies for the last two electrons. This gets pretty close to thermal energies of $k_B T = 0.025$ eV at room temperature. We also have a modified Bohr radius of

$$r_1 = \frac{a_0}{Z} \; \Rightarrow \; \frac{a_0}{1}\,\kappa$$

which for Si gives (0.053 nm)·12=0.64 nm and for Ge gives (0.053 nm)·16=0.85 nm. These last two electrons are delocalized from one atom because nearest neighbor distances are 0.23 nm for Si and 0.24 nm for Ge.