PHY 251 Spring 2008: homework problem set 8, due Thursday, April 10.

Serway problem 6.23

Answer: Because the potential goes to ∞ at $x = 0$, we need to have $\psi(x = 0) = 0$ so that we don't have a divergence at $U(x \leq 0) \cdot \psi(x \leq 0)$. In region I ($0 \leq x \leq L$) we have $U = 0$ so a valid wavefunction is (see p. 201)

$$\psi_I = A\sin(kx) \qquad \text{with} \qquad k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{\sqrt{2mE}}{\hbar}.$$

In region II ($x > L$) we are in a classically disallowed region so we will have an exponentially decaying wave function of the form (see Serway Eq. 6.21)

$$\psi_{II} = Be^{-\alpha(x-L)} \qquad \text{with} \qquad \alpha = \frac{\sqrt{2m(U-E)}}{\hbar}.$$

Our boundary conditions are

$$\psi_I(x = L) = \psi_{II}(x = L) \qquad \text{and} \qquad \frac{d\psi_I}{dx}|_{x=L} = \frac{d\psi_{II}}{dx}|_{x=L}$$

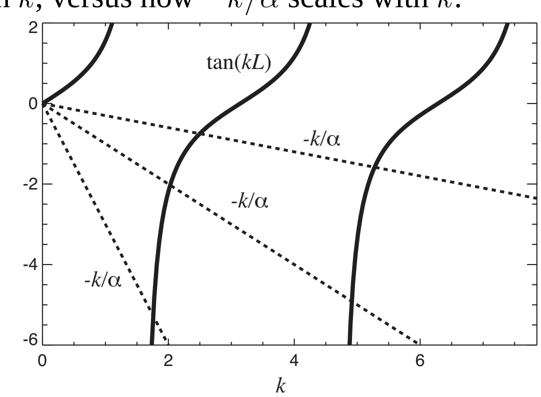
or

$$A\sin(kL) = Be^{-\alpha\cdot 0} = B \qquad \text{and} \qquad Ak\cos(kL) = -\alpha Be^{-\alpha\cdot 0} = -\alpha B$$

and if we divide these two equations we have

$$\frac{A\sin(kl)}{Ak\cos(kl)} = \frac{B}{-\alpha B} \qquad \Rightarrow \qquad \tan(kL) = -\frac{k}{\alpha}.$$

This does not give us a simple solution, but we can learn something from a graphical solution of how $\tan(kL)$ scales with k, versus how $-k/\alpha$ scales with k:



As you can see, no matter what slope $-1/\alpha$ we pick, we're going to have only discrete points where $\tan(kL)$ intersects with $-k/\alpha$. Once we identify a point k where such a solution occurs, we can find E from $k = \sqrt{2mE}/\hbar$ or better yet we can write $\tan(kL) = -k/\alpha$ as

$$\tan\left(\frac{L\sqrt{2mE}}{\hbar}\right) = -\sqrt{\frac{E}{U-E}}.$$

Serway problem 8.10

Answer: The orbital angular momentum is

$$|L| = \hbar\sqrt{\ell(\ell+1)} = (1.055 \times 10^{-34}\ \mathrm{J \cdot s})\sqrt{3(3+1)} = 3.65 \times 10^{-34}\ \mathrm{J \cdot s}$$

and the $\hat{z}$ component with $m_\ell = 3$ is

$$L_z = m_\ell \hbar = 3(1.055 \times 10^{-34}\ \mathrm{J \cdot s}) = 3.17 \times 10^{-34}\ \mathrm{J \cdot s}.$$

Serway problem 8.14

Answer:

a. For $n = 1$, only $\ell = 0$ and $m_\ell = 0$ is possible. For $n = 2$, we can have $\ell = 0$ and $m_\ell = 0$, or $\ell = 1$ and $m_\ell = [-1, 0, 1]$.

b. Li^{2+} has $Z = 3$ for a nuclear charge, so the Bohr model energies are

$$E = -13.6\frac{3^2}{1^2} = -122.4\ \mathrm{eV} \qquad \text{and} \qquad E = -13.6\frac{3^2}{2^2} = -30.6\ \mathrm{eV}$$

for $n = 1$ and $n = 2$, respectively.

Serway problem 8.16

Answer: Counting up from s to p to d, we count through $\ell = 0$ to $\ell = 1$ to $\ell = 2$. With $\ell = 2$, we have possible values of $L_z = [-2, -1, 0, 1, 2]\hbar$.

Serway problem 8.25

Answer: The most probable radius come finding the peak of $P = r^2|\psi(r)|^2$, which can be found by setting the derivative to zero. For the $2s$ state, this is

$$\begin{aligned} r_{prob} \Rightarrow \qquad 0 &= \frac{d}{dr}\left(r^2\left(2 - \frac{r}{a_0}\right)^2 e^{-r/a_0}\right) \\ &= \left[2r\left(2 - \frac{r}{a_0}\right)^2 + \left(2r^2\left(2 - \frac{r}{a_0}\right) + r^2\left(2 - \frac{r}{a_0}\right)^2\right)\left(\frac{-1}{a_0}\right)\right] e^{-r/a_0} \\ &= \frac{r}{(a_0)^3}\left(8(a_0)^3 - 16(a_0)^2 r + 8a_0 r^2 - r^3\right) e^{-r/a_0}, \end{aligned}$$

which has minima at $r = 0$ and $r = 2a_0$, and maxima at $(3 \pm \sqrt{5})a_0$. The maximum at $r = (3 + \sqrt{5})a_0$ is about 4 times the size of the other, so that for the $2s$ state, the most probable radius is $r = (3 + \sqrt{5})a_0 \approx 5.24a_0$.

For the $2p$ state,

$$\begin{aligned} r_{prob} \Rightarrow \qquad 0 &= \frac{d}{dr}\left(r^4 e^{-r/a_0}\right) \\ &= r^3\left(4 - \frac{r}{a_0}\right) e^{-r/a_0}, \end{aligned}$$

which has a minimum at $r = 0$ and a maximum at $r = 4a_0$, which is the same as the Bohr radius for $n = 2$.

Serway problem 8.26

Answer: The radial wavefunction integrals are

$$\begin{aligned} 2s\text{: } \int [R_{20}(r)]^2 \; r^2\,dr &= \int \left(\frac{Z}{2a_0}\right)^3 \left(2 - \frac{Zr}{a_0}\right)^2 e^{-Zr/a_0}\, r^2\,dr \\ 2p\text{: } \int [R_{21}(r)]^2 \; r^2\,dr &= \int \left(\frac{Z}{2a_0}\right)^3 \left(\frac{Zr}{\sqrt{3}a_0}\right)^2 e^{-Zr/2a_0}\, r^2\,dr \end{aligned}$$

Serway problem 9.5

Answer: If the atom has a speed v and travels through the field gradient over a distance d, the amount of time it is in the field gradient is given by $t = d/v$. During that time, it experiences a constant acceleration of $a_z = F_z/m$ so the net deflection distance z is given by

$$z = \frac{1}{2}a_z t^2 = \frac{1}{2}\frac{\mu_z}{m}\frac{dB_z}{dz}(\frac{d}{v})^2$$

with $\mu_z = g\mu_B m_s$ where $g = 2$ is described on the bottom of Serway p. 309. Solving for the field gradient dB_z/dz gives

$$\frac{dB_z}{dz} = 2z\frac{m}{2\mu_z}\frac{v^2}{d^2} = 2(10^{-3})\,\frac{(107.87\text{ amu})\cdot(1.66\times10^{-27}\text{ kg/amu})}{2\cot(9.274\times10^{-24})\cdot 1/2}\,\frac{100^2}{1^2} = 0.38\text{ T/m}$$

Serway problem 9.12

Answer: In spectroscopic notation, we add to our usual $1s$ or $2p$ type notation a subscript for $j = \ell + s$ where s is the total $\hat{z}$ axis angular momentum $m_\ell + m_s$. A) For $n = 7$, $\ell = 4$ (giving f), and $j = 9/2$ we have $s = j - \ell = 9/2 - 8/2 = 1/2$ so we have a $7f_{1/2}$ state. B) for $n = 6$ and $\ell = 5$ we have a $6g$ state with possible spin . . . NEED TO FINISH

Serway problem 9.21

Answer:

a. Oxygen has 8 electrons in the configuration $1s^2 2s^2 2p^4$

b. Orbitals are filled in energy order. To find how orbitals with the same energy are filled, we need to follow Hund's second and third rule (which have to be satisfied in numerical order):

   2. The lowest energy atomic state is the one which maximizes the total spin $S = \sum m_s$. This means shells of the same energy are filled with $m_s = +\frac{1}{2}$ first.
   3. The lowest energy atomic state is the one which maximizes the total angular momentum $L = \sum m_\ell$. This means if several constellations with maximum total spin S are possible, the shells with highest magnetic quantum numbers m_ℓ are filled first.

   This leads to the following sets of quantum numbers:

n	1	1	2	2	2	2	2	2
ℓ	0	0	0	0	1	1	1	1
m_ℓ	0	0	0	0	+1	0	-1	+1
m_s	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$

Extra problem 1: An electron is placed in an infinite potential box of width 0.2 nm. Calculate the ground state energy of the state. Then, assume that the potential of the box is not infinite but 2.0 eV. Calculate the $1/e$ tunneling width δ of the box, use it to generate a modified width, and calculate the approximate energy of the state with this modification.

Answer: With the infinite box, the ground state energy is

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2(hc)^2}{8(mc^2)L^2} = \frac{1^2(1240\text{ eV}\cdot\text{nm})^2}{8(511\times10^3\text{ eV})(0.2\text{ nm})^2} = 9.4\text{ eV}.$$

The tunneling is characterized by a width

$$\begin{aligned}\delta &= \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0-E)}} = \frac{hc}{\sqrt{8\pi^2mc^2(U_0-E)}} \\ &= \frac{1240\text{ eV}\cdot\text{nm}}{\sqrt{8\pi^2(511\times10^3\text{ eV})(9.4-2.0\text{ eV})}} = 0.072\text{ nm}\end{aligned}$$

If we then use this width to effectively increase the width of the square well, the energy is changed by

$$E \propto \frac{1}{L^2} \qquad \rightarrow \qquad E_{\text{new}} = E_{\text{old}}\left(\frac{L_{\text{old}}}{L_{\text{new}}}\right)^2 = (9.4\text{ eV})\left(\frac{0.2\text{ nm}}{0.2+2\cdot0.072\text{ nm}}\right)^2 = 3.2\text{ eV}.$$

Of course this approximation would be more accurate if the "widening" of the quantum well by tunneling would have been smaller. . .

Extra problem 2: What is the likelihood of a student tunneling through a wall that is 2 nm thick (about 10 atoms) and 2 meters high? Consider the student to have a mass of 80 kg, and to run at the wall with a velocity of 5 m/sec. Calculate the decline in the student's probability of being on the far side of the wall versus the near side. Of course another way for the student to go through such a thin wall is to rip through it. . .

Answer: The wavefunction amplitude declines by $\exp[-x/\delta]$ so the probably declines by $(\exp[-x/\delta])^2$. The tunneling distance δ is

$$\begin{aligned}\delta &= \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0-E)}} = \frac{h}{\sqrt{8\pi^2m(U_0-E)}} = \frac{h}{\sqrt{8\pi^2m(mgh-\frac{1}{2}mv^2)}} \\ &= \frac{h}{\sqrt{8\pi^2m^2(gh-\frac{1}{2}v^2)}} = 3.5\times10^{-37}\text{ m}.\end{aligned}$$

We therefore have $x/\delta = (2\times10^{-9}\text{ m})/(3.5\times10^{-37}\text{ m}) = 5.7\times10^{27}$ so $(\exp[-x/\delta])$ is an exceedingly small number. Now the lifetime of the universe is about

$$(13\times10^9\text{ years})(365\cdot24\cdot60\cdot60\,\frac{\text{sec}}{\text{year}}) = 4.1\times10^{17}\text{ seconds},$$

so even if the student were to hit the wall once per second over the lifetime of the universe the odds of tunneling would still be vanishingly small.