PHY 251 Spring 2008: homework problem set 7, due Thursday, April 3.

Serway problem 5.19

Answer: The proton mass is 938.3 MeV so a kinetic energy of only 1.0 MeV is very nonrelativistic. We then have $E_k = p^2/(2m)$ or

$$p = \sqrt{2mE_k} = \sqrt{2mc^2 E_k}/c = \sqrt{2(938.3\text{ MeV})(1.0\text{ MeV})}/c = 43.3\text{ MeV}/c$$

and 5% of that gives $\Delta p = 2.17$ MeV/c. The uncertainty principle gives

$$\Delta x = \frac{\hbar}{2\Delta p} = \frac{hc}{4\pi(\Delta p)c} = \frac{1240 \times 10^{-9}\text{ eV}\cdot\text{m}}{4\pi(2.17 \times 10^6\text{ eV}/c)c} = 4.55 \times 10^{-14}\text{ m}.$$

Serway problem 5.25

Answer: From the energy-time version of the uncertainty principle, we can associate a mean lifetime τ as giving an energy width of

$$\Delta E \approx \frac{\hbar}{2\tau} = \frac{h}{4\pi\tau} = \frac{(6.63 \times 10^{-34}\text{ J}\cdot\text{s})(1\text{ eV}/1.6 \times 10^{-19}\text{ J})}{4\pi \cdot 10^{-10}\text{ s}} = 3.3 \times 10^{-6}\text{ eV}.$$

This is too small an energy blur to measure with a detector with 5 eV energy resolution.

Serway problem 5.26

Answer: The full-width half-maximum distribution is about 5 bins across, so we'll characterize the 1σ error as 2.5 bins of width 25 MeV/c^2 so $\Delta E = (2.5)(25) = 68$ MeV using $E = mc^2$. The lifetime is thus

$$\Delta t = \frac{\hbar}{2\Delta E} = \frac{6.58 \times 10^{-16}\text{ eV}\cdot\text{sec}}{2(68 \times 10^6\text{ eV})} = 4.4 \times 10^{-24}\text{ seconds}$$

Serway problem 5.33

Answer: We have $\Delta E \cdot \Delta t \simeq \hbar/2$, so

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{6.582 \times 10^{-16}\text{ eV}\cdot\text{sec}}{2 \cdot 8.7 \times 10^{-17}\text{ sec}} = 3.8\text{ eV}.$$

Because $E = mc^2$, we have identical ratios $\Delta E/E = \Delta m/m = (3.8/135 \times 10^6) = 2.8 \times 10^{-8}$.

Serway problem 6.9

Answer: The energy of the photon is given by

$$E_\gamma = \frac{\pi^2\hbar^2}{2mL^2} \cdot (2^2 - 1^2) = \frac{3\pi^2 h^2}{8\pi^2 mL^2} = \frac{3}{8}\,\frac{(hc)^2}{mc^2\,L^2} = \frac{3}{8}\,\frac{(1240 \times 10^{-9}\text{ eV}\cdot\text{m})^2}{(938.3 \times 10^6\text{ eV})\cdot(10^{-14}\text{ m})^2}$$

or $E_\gamma = 6.1 \times 10^6$ eV or 6.1 MeV. This is known as a gamma ray (a photon with an energy of greater than about 100 keV).

Serway problem 6.12

Answer: The photon energy $E_\gamma = (1240/694.3) = 1.786$ eV must be given by

$$\begin{aligned} E_\gamma = &= \frac{\pi^2\hbar^2}{2mL^2}(2^2 - 1^2) \\ L^2 &= \frac{3\pi^2 h^2}{8\pi^2 m E_\gamma} = \frac{3(hc)^2}{8mc^2\, E_\gamma} \\ L &= \frac{\sqrt{3}\,(hc)}{\sqrt{8mc^2\, E_\gamma}} = \frac{\sqrt{3}\,(1240 \times 10^{-9}\text{ eV}\cdot\text{ m})}{\sqrt{8\cdot(511\times 10^3\text{ eV})\cdot(1.786\text{ eV})}} = 7.9\times 10^{-10}\text{ m} \end{aligned}$$

or 0.79 nm.

Serway problem 6.16

Answer: The probability p of being in the region between x_1 and x_2 is given by

$$\begin{aligned} p &= \int_{x_1}^{x_2} |\psi(x)^2|\,dx = \int_{x_1}^{x_2}\left[\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right]^2 dx \\ &= \frac{2}{L}\frac{1}{k}\int_{x_1}^{x_2}\sin^2(kx)\,d(kx) \qquad \text{with} \qquad k \equiv \frac{n\pi}{L} \\ &= \frac{2}{kL}\left[\frac{1}{2}kx - \frac{1}{4}\sin(2kx)\right]\Big|_{x_1}^{x_2} \\ &= \frac{2}{n\pi}\left[\frac{n\pi}{2L}(x_2 - x_1) - \frac{1}{4}(\sin(\frac{2n\pi x_2}{L}) - \sin(\frac{2n\pi x_1}{L}))\right] \end{aligned}$$

With a well of width 0.300 nm, and $n = 1$ (ground state), the probability of being within 0.100 nm of the left-hand wall corresponds to $x_2 = 0.100$, $x_1 = 0$, and $L = 0.300$. (Note that because we are always dealing with ratios of x/L we don't need to worry about the exact units). In this case the probability is

$$p = \frac{2}{\pi}\left[\frac{\pi}{2}\,\frac{0.1 - 0.0}{0.3} - \frac{1}{4}(\sin(\frac{2\pi\cdot 0.1}{0.3}) - \sin(0))\right] = 0.196$$

If we instead use $n = 100$ we find

$$p = \frac{2}{100\pi}\left[\frac{100\pi}{2}\,\frac{0.1 - 0.0}{0.3} - \frac{1}{4}(\sin(\frac{200\pi\cdot 0.1}{0.3}) - \sin(0))\right] = 0.332$$

which approaches the classical limit of having the particle in this region (1/3)=0.333 of the time.

Serway problem 6.19

Answer: For a one dimensional infinite square well, we have

$$\begin{aligned} E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2} \\ \text{or} \qquad n &= \frac{L\sqrt{8mE}}{h} = \frac{0.0100\sqrt{8\cdot 1.00\times 10^{-3}\cdot 1.00\times 10^{-3}}}{6.626\times 10^{-34}} = 4.3\times 10^{28}. \end{aligned}$$

We used all MKS units to get the units right. The quantum number is so high that we can't see individual quantum excitations. How much energy is required to increase the quantum number n? We can find this out from dE/dn or

$$dE = 2\frac{nh^2}{8mL^2}\,dn = \frac{h^2}{4mL^2}\,n\,dn$$

or using $dn = 1$ for one excitation we have

$$\Delta E = \frac{(6.626 \times 10^{-34})^2}{4 \cdot 1.00 \times 10^{-3} \cdot (0.0100)^2}\,(4.3 \times 10^{28})\,(1) = 4.7 \times 10^{-34} \text{ Joules}$$

or 2.9×10^{-13} eV or a temperature of

$$T = \frac{E}{k_B} = \frac{2.9 \times 10{-}13 \text{ eV}}{8.617 \times 10^{-5} \text{ ev/K}} = 3.4 \times 10^{-9} \text{ K}.$$

In other words, it would be a really cold day in hell before you saw such an excitation.

Serway problem 6.25

Answer: Classically, all of a harmonic oscillator's energy is potential energy when it is at max amplitude, so the energy is

$$E_n = \frac{1}{2}kA_n^2$$

where $k = m\omega^2$. The energies of the quantum mechanical harmonic oscillator are given by $E = (n + 1/2)\hbar\omega$, so if we substitute that into the above we have

$$\begin{aligned}
(n + \frac{1}{2})\hbar\omega &= \frac{1}{2}kA_n^2 \\
(2n+1)\hbar\omega &= m\omega^2 A_n^2 \\
(2n+1)\frac{\hbar}{m\omega} &= A_n^2 \\
A_n &= \sqrt{\frac{(2n+1)\hbar}{m\omega}}
\end{aligned}$$

which is what we wanted to verify.

Serway problem 6.32

Answer: The lowest energy solution wavefunction is $\psi = C_0 e^{-\alpha x^2}$ so we have

$$\langle x \rangle = \int_{-\infty}^{\infty} x\,(C_0 e^{-\alpha x^2})^2\,dx = C_0^2 \int_{-\infty}^{\infty} x\,e^{-2\alpha x^2}\,dx.$$

If we set $y \equiv e^{-2\alpha x^2}$ we have $dy = -4\alpha x\,e^{-2\alpha x^2}\,dx$ so the above integral becomes

$$\langle x \rangle = -\frac{C_0^2}{4\alpha} \int_{x=-\infty}^{x=\infty} dy = -\frac{C_0^2}{4\alpha} e^{-2\alpha x^2}|_{-\infty}^{\infty} = 0$$

because $e^{-2\alpha\infty} \rightarrow 0$ no matter what the value of α is. For $\langle x^2 \rangle$ we have

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \, (C_0 e^{-\alpha x^2})^2 \, dx = C_0^2 \int_{-\infty}^{\infty} x^2 \, e^{-2\alpha x^2} \, dx$$

and if we make the substitution $a \equiv 2\alpha$ we have

$$\langle x^2 \rangle = C_0^2 \int_{-\infty}^{\infty} x^2 \, e^{-ax^2} \, dx = C_0^2 \, \frac{1}{4a} \sqrt{\frac{\pi}{a}} = C_0^2 \, \frac{1}{8\alpha} \sqrt{\frac{\pi}{2\alpha}}.$$

Now let's use the result $\alpha = m\omega/(2\hbar)$ from Serway Eq. 6.27, and $C_0^2 = \sqrt{(m\omega)/(\pi\hbar)}$ from Serway Example 6.10 to find

$$\langle x^2 \rangle = C_0^2 \frac{1}{8\alpha} \sqrt{\frac{\pi}{2\alpha}} = \frac{\sqrt{m\omega}}{\sqrt{\pi\hbar}} \frac{2\hbar}{8m\omega} \sqrt{\frac{2\pi\hbar}{2m\omega}} = \frac{\hbar}{4m\omega}.$$

Let's look at the units of this. The units of $\hbar$ are of energy times time, or kg·m^2/s. The units of $m\omega$ is in kg/s. Therefore the units of $\hbar/(4m\omega)$ are meters squared as expected for $\langle x^2 \rangle$. Finally, we can use Serway Eq. 6.34 to find

$$\Delta x = \sqrt{\langle x^2 \rangle - (\langle x \rangle)^2} = \sqrt{\frac{\hbar}{4m\omega} - (0)^2} = \frac{1}{2} \sqrt{\frac{\hbar}{m\omega}}.$$