

PHY 251 Spring 2008: homework problem set 6, due Thursday, March 27.

**Serway problem 4.33**

*Answer:* The reduced mass of the lead nucleus with an orbiting muon is

$$m_r = \frac{m_\mu m_{\text{Pb}}}{m_\mu + m_{\text{Pb}}} \simeq m_\mu = 207m_e \quad \text{because} \quad m_{\text{Pb}} \gg m_\mu.$$

Next, we need to use the reduced mass version of the Bohr model that we worked out in the lecture notes (slide 23 of lecture 10), or

$$E_n = -\frac{Z^2 m_r e^4}{n^2 8h^2 \epsilon_0^2} = -\frac{Z^2}{n^2} E_0 \frac{m_r}{m_e}.$$

Since  $m_r \simeq m_\mu$  and since  $Z = 82$ , we have

$$E_1 = -\frac{Z^2}{n^2} E_0 \frac{m_r}{m_e} = -\frac{(82)^2}{1^2} (13.6 \text{ eV}) \frac{207m_e}{m_e} = 1.89 \times 10^7 \text{ eV}.$$

The Bohr radius with the reduced mass taken into account is (see slide 21 of lecture 10)

$$r_n = \frac{n^2}{Z} \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} \frac{m_e}{m_r} = \frac{n^2}{Z} a_0 \frac{m_e}{m_r}$$

with  $a_0 = 0.053 \text{ nm}$ , which in this case gives

$$r_1 = \frac{1^2}{82} (0.053 \text{ nm}) \frac{m_e}{207m_e} = 3.1 \times 10^{-6} \text{ nm} = 3.1 \times 10^{-15} \text{ m}$$

which is the range of the size of the Pb nucleus!

**Serway problem 4.34**

*Answer:* Again we have  $m_\mu = 207m_e$ . To do this problem we certainly need to use the reduced mass version of the Bohr model that we worked out in the lecture notes (slide 23 of lecture 10), or

$$E_n = -\frac{Z^2 m_r e^4}{n^2 8h^2 \epsilon_0^2} = -\frac{Z^2}{n^2} E_0 \frac{m_r}{m_e}.$$

So what's the reduced mass for this atom? Use  $m_\mu = 207m_e$  and the proton and neutron masses in  $\text{MeV}/c^2$  given in Appendix 1 of Serway:

$$m_r = \frac{m_\mu (m_p + m_n)}{m_\mu + (m_p + m_n)} = \frac{207 \cdot m_e (938.3 + 939.6)}{207 \cdot 0.511 + (938.3 + 939.6)} = 196.0 m_e$$

where we left  $m_e$  in there as a multiplier in the numerator but put in a numerical value in the denominator. We then have Bohr model energies of

$$E_n = -\frac{Z^2}{n^2} E_0 \frac{m_r}{m_e} = -\frac{1^2}{n^2} (13.61 \text{ eV}) \frac{196.0 m_e}{m_e} = \frac{2668 \text{ eV}}{n^2}.$$

Therefore the energy difference between  $n = 1$  and  $n = 2$  is  $(1/1^2 - 1/2^2) = 3/4$  of 2668 eV, or 2001 eV, and the wavelength of the emitted photon is  $\lambda = hc/E = 1240/2001 = 0.62 \text{ nm}$ .

**Serway problem 4.43**

*Answer:* The atom mass is  $m = 10^{-25}$  kg, and it has an initial velocity of  $v_1$ . The photon has an energy of  $E = hc/\lambda = 1240/500 = 2.48$  eV. If we say that the atom is moving in the  $+x$  direction and the photon comes in head-on in the  $-x$  direction, conservation of momentum tells us about the momentum of the atom after the collision:

$$\begin{aligned}mv_1 - \frac{E}{c} &= mv_2 \\mv_2 - mv_1 &= -\frac{E}{c} \\\Delta v = v_2 - v_1 &= -\frac{E}{mc} =\end{aligned}$$

If it takes  $10^{-8}$  seconds before the atom deexcites by spontaneous emission, then the atom is next able to absorb a  $\lambda = 500$  nm photon at that same time later. As a result, the acceleration is

$$a = -\frac{\Delta v}{\Delta t} = \frac{E/mc}{\Delta t} = -\frac{E}{mc \Delta t} = -\frac{(2.48 \text{ eV}) \cdot (1.6 \times 10^{-19} \text{ J/eV})}{(10^{-25} \text{ kg}) \cdot (2.99 \times 10^8 \text{ m/s}) \cdot (10^{-8} \text{ s})} = -1.3 \times 10^6 \text{ m/s}^2$$

Because the spontaneously emitted photons have no preferred direction, they produce no preferred momentum kick; that is, they average out to zero acceleration. Now that we know the acceleration, we can figure out that the atoms slow down over a distance of

$$\begin{aligned}v_f^2 &= v_0^2 + 2a(x - x_0) \\(x - x_0) &= \frac{v_f^2 - v_0^2}{2a} = \frac{0^2 - (10^3 \text{ m/s})^2}{2(-1.3 \times 10^6)} = 0.38 \text{ m}\end{aligned}$$

**Serway problem 4.44**

*Answer:* In the Bohr model, the wavelengths corresponding to state transitions are given by

$$\lambda = \frac{hc}{\Delta E} = \frac{hc}{Z^2 E_0} \frac{1}{1/n_f^2 - 1/n_i^2} = a \frac{hc}{Z^2 E_0}$$

where  $n_i > n_f$ . Regarding the factor  $a = 1/(1/n_f^2 - 1/n_i^2)$ , the initial state can range from  $n_i \rightarrow \infty$  to  $n_i = n_f + 1$ . Therefore the ratio of wavelengths for the series with the same final state is

$$\begin{aligned}\frac{\lambda_{\text{shortest}}}{\lambda_{\text{longest}}} &= \frac{1/n_f^2 - 1/(n_f + 1)^2}{1/n_f^2 - 1/\infty} = \frac{1 - n_f^2/(n_f^2 + 2n_f - 1)}{1 - 0} \\&= 1 - n_f^2/(n_f^2 + 2n_f + 1) = 1 - \frac{1}{1 + 2/n_f + 1/n_f^2}\end{aligned}$$

which has values of 0.75 for  $n_f = 1$ , 0.55 for  $n_f = 2$ , 0.44 for  $n_f = 3$ , 0.36 for  $n_f = 4$ , 0.31 for  $n_f = 5$ , and so on. Since in our case we have a wavelength ratio of  $22.8/63.3 = 0.36$ ,

we see that we have  $n_f = 4$ . We can then find  $Z$  using  $\lambda = 22.8 \text{ nm}$  for the  $n_i \rightarrow \infty$  and  $n_f = 4$  transition as

$$\lambda = \frac{hc}{\Delta E} = \frac{hc}{Z^2 E_0/n_f^2} \quad \rightarrow \quad Z = n_f \sqrt{\frac{hc}{\lambda E_0}} = 4 \sqrt{\frac{1240 \text{ eV} \cdot \text{nm}}{(22.8 \text{ nm}) \cdot (13.6 \text{ eV})}} = 8.00$$

so it's an Oxygen atom we're talking about.

### Serway problem 5.2

*Answer:* In both cases the kinetic energy is pretty small compared to  $mc^2$  so we can use the classical result of  $E_k = p^2/(2m)$  to write

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{2mc^2 E_k}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2 \cdot 511 \times 10^3 \cdot [5 \times 10^1, 5 \times 10^4]}}$$

or  $\lambda = [0.17, 0.0055] \text{ nm}$ .

For the sake of argument, let's look at the relativistic correction to the above result. The kinetic energy is  $E_k = (\gamma - 1)mc^2$  or  $\gamma = 1 + E_k/(mc^2)$  and then

$$\lambda = \frac{h}{p} = \frac{hc}{\gamma m v c} = \frac{hc}{\gamma \beta mc^2}.$$

For the two different electron energies we have  $\gamma = 1 + E_k/(mc^2)$  and

$$\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - (1 + E_k/(mc^2))^{-2}} \simeq \sqrt{1 - (1 - 2E_k/(mc^2))} = \sqrt{\frac{2E_k}{mc^2}}$$

and finally

$$\gamma \beta \simeq \left(1 + \frac{E_k}{mc^2}\right) \sqrt{\frac{2E_k}{mc^2}}$$

giving

$$\lambda = \frac{hc}{\gamma \beta mc^2} \simeq \frac{hc}{\sqrt{\frac{2E_k}{mc^2}} mc^2 \left(1 + \frac{E_k}{mc^2}\right)} \simeq \frac{hc}{\sqrt{2mc^2 E_k}} \left(1 - \frac{E_k}{mc^2}\right)$$

so we see that the relativistic correction to the de Broglie wavelength is small: it goes like  $E_k/mc^2$ .

### Serway problem 5.3

*Answer:* We have

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.63 \times 10^{-34}}{(74)(5)} = 1.8 \times 10^{-36} \text{ m}.$$

**Serway problem 5.10**

*Answer:* The diameter of the hydrogen atom (with  $Z = 1$  and  $n = 1$ ) is  $2a_0$ , and we know that the ground state of the hydrogen atom involves non-relativistic electrons (we found  $v = \alpha c \simeq c/137$ ) so  $E_k = p^2/(2m)$  or  $p = \sqrt{2mE_k}$ . If we set this equal to the de Broglie wavelength, we have

$$\begin{aligned} 2a_0 &= \lambda = \frac{h}{p} = \frac{hc}{\sqrt{2mc^2E_k}} \\ 4a_0^2 &= \frac{(hc)^2}{2mc^2E_k} \\ E_k &= \frac{(hc)^2}{8mc^2a_0^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8 \cdot (511 \times 10^3 \text{ eV}) \cdot (0.053 \text{ nm})^2} = 134 \text{ eV} \end{aligned}$$

This energy is much larger than the 13.6 eV energy of the hydrogen ground state. Of course the construction for the ground state of hydrogen is to make the electron wavelength equal to the circumference of the orbit.

**Serway problem 5.14**

*Answer:* These are low-energy electrons, so they're in the classical limit and the kinetic energy is  $E_k = p^2/2m$  giving  $p = \sqrt{2mE_k}$ . The de Broglie wavelength is then

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}} = \frac{hc}{\sqrt{2m_e c^2 E_k}}.$$

If we say that the crystal planes form a simple grating at the top surface (because the electrons don't travel very far in the material), and the electron beam scatters off of that grating, then we have a simple, planar grating equation (as opposed to a volume or Bragg grating equation of  $d \sin \theta = n\lambda$ ). The simple grating equation is

$$\sin \theta = \frac{n\lambda}{d} = \frac{nhc}{d\sqrt{2(m_e c^2)E_k}}.$$

Well, let's move on to part B. If we have two angles for successive peaks  $n$  and  $n + 1$ , we can write

$$\sin \theta_{n+1} - \sin \theta_n = (n + 1) \frac{hc}{d\sqrt{2(mc^2)E_k}} - n \frac{hc}{d\sqrt{2(mc^2)E_k}} = \frac{hc}{d\sqrt{2(mc^2)E_k}}$$

We can then solve for the atomic spacing  $d$ :

$$\begin{aligned} d &= \frac{hc}{(\sin \theta_{n+1} - \sin \theta_n) \sqrt{2(mc^2)E_k}} \\ &= \frac{1240 \text{ eV} \cdot \text{nm}}{(\sin 54.9^\circ - \sin 24.1^\circ) \sqrt{2 \cdot 511 \times 10^3 \text{ eV} \cdot 100 \text{ eV}}} = (0.30 \text{ nm}) \end{aligned}$$

or  $d = 3 \text{ \AA}$  which is a reasonable inter-atomic distance or lattice spacing in solids. By the way, we can also infer the diffraction orders  $n$  from realizing that  $\sin \theta = (\lambda/d)n$  so if we look at the ratio of the two angles of

$$\frac{\sin \theta_A}{\sin \theta_B} = \frac{(\lambda/d)n_A}{(\lambda/d)n_B} = \frac{n_A}{n_B}$$

which in this case is  $\sin 54.9^\circ / \sin 24.1^\circ = 0.818 / .408 \simeq 2/1$  so we can guess that  $n_A = 2$  and  $n_B = 1$ .

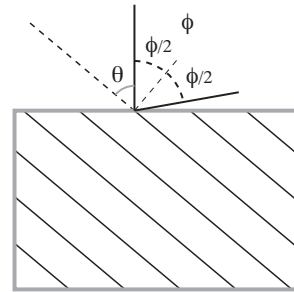
It's worth comparing the above result with a volume or Bragg grating result (see right). In this case, the net deflection angle is  $\phi$ , and the Bragg angle  $\theta$  is given by  $\theta = \pi/2 - \phi/2$ . Therefore we can write Bragg's law as

$$2d \sin \theta = 2d \sin(\pi/2 - \phi/2) = 2d \cos(\phi/2) = n\lambda.$$

We then have

$$\lambda = \frac{2d \cos(\phi/2)}{n} = \frac{hc}{\sqrt{2m_e c^2 E_k}}$$

$$\text{or } 2 \cos(\phi/2) = \frac{nhc}{d\sqrt{2m_e c^2 E_k}}.$$



The moral of the story is that one gets significantly different results from scattering from a simple grating on a surface, or scattering from a volume grating.

### Serway problem 6.5

*Answer:* We have  $\psi(x) = Axe^{-x^2/L^2}$  for a particle with zero energy. The derivatives of  $\psi$  are

$$\begin{aligned} \frac{\partial}{\partial x} Axe^{-x^2/L^2} &= Ae^{-x^2/L^2} + Ax\left(-\frac{2x}{L^2}\right)e^{-x^2/L^2} = \psi \left( \frac{1}{x} - 2\frac{x}{L^2} \right) \\ \frac{\partial^2}{\partial x^2} Axe^{-x^2/L^2} &= \frac{\partial}{\partial x} \left[ Ae^{-x^2/L^2} - 2A\left(-\frac{x^2}{L^2}\right)e^{-x^2/L^2} \right] \\ &= A\left(-2\frac{x}{L^2}\right)e^{-x^2/L^2} + A(-2) \end{aligned}$$

If we use  $E = 0$  in the Schrödinger equation we have

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + U\psi &= E\psi = 0 \\ U\psi &= \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi \\ UAxe^{-x^2/L^2} &= a \end{aligned}$$

(Must finish solution. . .)

**Serway problem 6.6**

*Answer:* If  $U = 0$  the Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E\psi$$

so we need to take the second derivative of the wavefunction:

$$\begin{aligned} \frac{d\psi}{dx} &= \frac{d}{dx} (A \cos(kx) + B \sin(kx)) = -kA \sin(kx) + kB \cos(kx) \\ \frac{d}{dx} (-kA \sin(kx) + kB \cos(kx)) &= -k^2 A \cos(kx) - k^2 B \sin(kx) = (-k^2)\psi \end{aligned}$$

Then the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m} (-k^2)\psi = E\psi$$

so this choice of wavefunction satisfies the Schrödinger equation. Furthermore we find that the energy is given by

$$E = \frac{\hbar^2}{2m} k^2 = \frac{h^2}{(4\pi^2)2m} \frac{4\pi^2}{\lambda^2} = \frac{h^2}{2m} \frac{p^2}{h^2} = \frac{p^2}{2m}$$

and since  $p^2/(2m) = (1/2)mv^2$  we have found that the particle's energy is its kinetic energy!