PHY 251 Spring 2008: homework problem set 5, due Thursday, March 6.

Serway problem 4.3

Answer: In the apparatus, we have

$$F_y = q\frac{V}{d} = m\frac{\Delta v_y}{\Delta t}$$

and we also have $v_x = \ell/\Delta t$ or $\Delta t = \ell/v_x$. Therefore we can write

$$\Delta v_y = \frac{qV\,\Delta t}{md} = \frac{qV}{md}\frac{\ell}{v_x}.$$

We can then figure out the angle θ:

$$\tan\theta = \frac{v_y}{v_x} = \frac{qV\ell}{mdv_x^2} \qquad \text{so} \qquad \frac{q}{m} = \frac{dv_x^2\tan\theta}{V\ell}.$$

Now from the magnetic field cancelling the electric field we can say

$$\frac{qV}{d} = qv_xB \qquad \text{so} \qquad v_x = \frac{V}{Bd} = \frac{2000}{4.57\times10^{-2}\cdot 0.02} = 2.19\times10^6 \text{ m/s}$$

which means relativistic effects are small ($\gamma = 1.000\,027$). We can also substitute this result for velocity into our expression for q/m:

$$\frac{q}{m} = \frac{d}{V\ell}\frac{V^2}{B^2d^2}\tan\theta = \frac{V}{ldB^2}\tan\theta = \frac{2000}{0.10\cdot 0.02\cdot(4.57\times10^{-2})^2}\tan(0.2) = 9.7\times10^7 \text{ C/kg}.$$

A proton has $q/m = 1.6\times10^{-19}/1.67\times10^{-27} = 9.7\times10^7$ so it seems like we have a good candidate. . .

Serway problem 4.4

Answer: This is a bit like the example we did around slide 20 of Lecture 8. The electric field the electron experiences when it travels between the two plates is $E_y = V/d$, and the acceleration it experiences is $a = F/m = qE/m = qV/(md)$. Since we expect $v_y \ll v_x$, we can say that the time that the electron experiences this acceleration is $\Delta t = \ell/v_x$. Consequently the electron receives a velocity kick in the $\hat{y}$ direction of

$$v_y = a_y\cdot\Delta t = \frac{qV}{md}\cdot\frac{\ell}{v_x}$$

so that its angle θ leaving the field region is $\theta = v_y/v_x = y_2/D$ (the latter by geometry in the small angle limit) or

$$\theta = \frac{y_2}{D} = \frac{v_y}{v_x} = \frac{qV}{md}\frac{\ell}{v_x^2}$$

which can be solved for y_2 to give

$$y_2 = \frac{qVD\ell}{mdv_x^2}.$$

We can calculate y_1 from the distance traveled under constant acceleration or

$$y_1 = \frac{1}{2}a(\Delta t)^2 = \frac{1}{2}\frac{qV}{md}(\frac{\ell}{v_x^2})^2 = \frac{qV\ell^2}{2mdv_x^2}.$$

Now since $y = y_1 + y_2$ we have

$$\begin{aligned} y &= y_1 + y_2 = \frac{qV\ell^2}{2mdv_x^2} + \frac{qV\,D\ell}{mdv_x^2} = \frac{q}{m}\frac{V\ell}{dv_x^2}(D + \ell/2) \\ \frac{q}{m} &= \frac{ydv_x^2}{V\ell(D+\ell/2)}. \end{aligned}$$

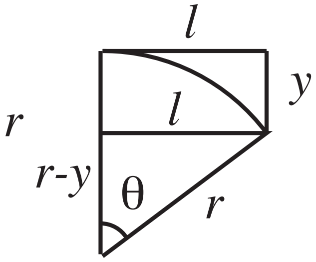
Serway problem 4.5

Answer: First consider the velocity selector:

$$qE = q\frac{V}{d} = qvB \qquad \text{so} \qquad v = \frac{V}{Bd}.$$

Now consider the geometrical problem: we have

$$\begin{aligned} r^2 &= l^2 + (r-y)^2 = l^2 + r^2 + y^2 - 2ry \\ 2ry &= l^2 + y^2 \\ r &= \frac{l^2+y^2}{2y} \end{aligned}$$



which is what we were asked to prove.
Now the difference in effects between relativistic and nonrelativistic forces can be expressed in terms of finding a value of γ other than 1 in the following expression:

$$\begin{aligned} \gamma m\frac{v^2}{r} &= qvB \\ \gamma &= \frac{qBr}{mv} = \frac{qB^2rd}{mV} = \frac{qB^2d}{mV}\frac{l^2+y^2}{2y} \\ &= \frac{1.602\times10^{-19}\cdot 0.0177^2\cdot 2.51\times10^{-4}}{9.11\times10^{-31}\cdot 1060}\frac{0.0247^2+0.0024^2}{2\cdot 0.0024} = 1.67 \end{aligned}$$

Since this is noticeably different from $\gamma = 1.00$, relativistic effects are crucial for a correct explanation of this experiment.

Serway problem 4.8

Answer: Our answer will be based on Serway Eq. 4.16 of

$$\Delta n = \frac{k^2Z^2e^4NnA}{4R^2(\frac{1}{2}m_\alpha v_\alpha^2)^2\sin^4(\phi/2)}$$

A. If we compare various angles we have a ratio

$$\begin{aligned}
\frac{\Delta n \text{ (new)}}{\Delta n \text{ (original)}} &= \frac{1/\sin^4(\phi_{\text{new}}/2)}{1/\sin^4(\phi_{\text{original}}/2)} \\
\Delta n \text{ (new)} &= \Delta n \text{ (original)} \frac{1 \sin^4(\phi_{\text{new}}/2)}{1/\sin^4(\phi_{\text{original}}/2)} \\
&= \Delta n \text{ (original)} \frac{\sin^4(\phi_{\text{original}}/2)}{\sin^4(\phi_{\text{new}}/2)} \\
&= \frac{100 \text{ particles}}{\text{minute}} \frac{\sin^4(20^\circ/2)}{\sin^4(40^\circ/2)} = 6.65
\end{aligned}$$

followed by $60^\circ \to 1.45$, $80^\circ \to 0.53$, and $100^\circ \to 0.26$.
B. If we consider the kinetic energy $E_k = \frac{1}{2} m_\alpha v_\alpha^2$, we see that the scaling is

$$\begin{aligned}
\frac{\Delta n \text{ (new)}}{\Delta n \text{ (original)}} &= \frac{1/(E_{k,\text{new}})^2}{1/(E_{k,\text{original}})^2} \\
\Delta n \text{ (new)} &= \Delta n \text{ (original)} \Big(\frac{E_{k,\text{original}}}{E_{k,\text{new}}}\Big)^2 = \frac{100 \text{ particles}}{\text{minute}} \Big(\frac{1}{2}\Big)^2 = 25
\end{aligned}$$

C. When changing from gold to copper, we need to worry about the nuclear charge Z, and the number of nuclei per area N (which for a foil of thickness t is

$$\frac{N \text{ atoms}}{\text{area}} = \frac{\rho \text{ (g/cm}^3) \cdot t \text{ (cm)} \cdot N_A \text{ (atoms/mol)}}{A \text{ (g/mol)}}$$

of which N_A is a constant and the thickness t is assumed to be the same for the two foils). As a result, the scaling from gold (old) to copper (new) goes as

$$\begin{aligned}
\frac{\Delta n \text{ (new)}}{\Delta n \text{ (original)}} &= \frac{\rho \text{ (new)} \cdot Z \text{ (new)}/A \text{ (new)}}{\rho \text{ (old)} \cdot Z \text{ (old)}/A \text{ (old)}} \\
&= \frac{8.9 \cdot 29/63.546}{19.3 \cdot 79/196.97} = 0.525
\end{aligned}$$

so the count rate would change from 100 particles/minute to 52.5 particles/minute.

Serway problem 4.9

Answer: The electrostatic force is $F = kq_1q_2/r^2$ so the work needed to bring the charge of an α particle from infinitely far away to a distance r from the nucleus is

$$\begin{aligned}
W &= \int_r^\infty F\, dx = kq_1q_2 \int_r^\infty \frac{1}{r'^2}\, dr' = kq_1q_2 \left(\frac{-1}{r'}\right) |_r^\infty \\
&= kq_1q_2 \left[\frac{-1}{\infty} - \frac{-1}{r}\right] = \frac{q_1q_2}{4\pi\epsilon_0} \frac{1}{r}.
\end{aligned}$$

When this equals the kinetic energy, the particle will stop and turn around or be deflected to the side, and Rutherford's expression (which is based solely on Coulomb repulsion) will hold. If instead the particle experiences some other force such as a nuclear binding force at this distance, then there will be deviations from Rutherford's law. So, what we want to do is to set the energy calculated above to be equal to the kinetic energy E_k of the α particle and solve for the distance r:

$$\begin{aligned} E_k &= \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r} \\ r &= \frac{q_1 q_2}{4\pi\epsilon_0 E_k} = \frac{(2e)(Ze)}{4\pi\epsilon_0 E_k} \\ &= \frac{2\cdot 29\cdot(1.602\times 10^{-19})^2}{4\pi\cdot 8.85\times 10^{-12}\cdot(13.9\times 10^6\text{ eV})\cdot(1.602\times 10^{-19}\text{ J/eV})} = 6.01\times 10^{-15}\text{ meters} \end{aligned}$$

where we have been careful to use mks units throughout to get an answer in mks units.

Serway problem 4.13

Answer: The general Bohr formula is

$$\Delta E = \frac{hc}{\lambda} = |E_{n_i} - E_{n_f}| = Z^2 E_0 \left|\frac{1}{n_i^2} - \frac{1}{n_f^2}\right|$$

or, with $Z = 1$,

$$\frac{hc}{\lambda\, E_0} = \frac{1240\text{ eV}\cdot\text{nm}}{(102.6\text{ nm})\cdot(13.60\text{ eV})} = 0.889 = \left|\frac{1}{n_i^2} - \frac{1}{n_f^2}\right|.$$

The Lyman series involves $n_i = 1$ (for absorption, or $n_f = 1$ for fluorescence) and the values for the right hand side of the above expression are 0.889, 0.938. . . for $n_f = 2, 3, \ldots$. That is, the only value of n_f that works is $n_f = 2$. If we were to make $n_i = 2$ or higher, we'd have $1/2^2 = 0.25$ or lower for the first term on the right hand side and we always must have the other state be a higher state number so there's no way we'd ever get a value for $|1/n_i^2 - 1/n_f^2|$ as high as 0.889 for any other combination but [1,2].

Serway problem 4.14

Answer: The radius of an electron's orbit in the Bohr model is

$$r = a_0 \frac{n^2}{Z} \qquad \text{with} \qquad a_0 = \frac{\hbar^2}{m_e k e^2} \qquad \text{and} \qquad k = \frac{1}{4\pi\epsilon_0}.$$

The Bohr model (and also the Schrödinger equation, which we will soon come to) is done in the classical limit, so we can say that the kinetic energy is $(1/2)mv^2$. In examining Serway Eq. 4.25 we find that the net binding energy of an electron is given by the potential plus the kinetic energy, or

$$E = K + U = \frac{ke^2}{2r} - \frac{ke^2}{r} = -\frac{ke^2}{2r}$$

so that the kinetic energy is the negative of the total energy. Thus we can say that

$$\begin{aligned}
\frac{ke^2}{2r} &= \frac{1}{2}m_e v^2 \\
\frac{ke^2}{2}\frac{m_e ke^2}{\hbar^2}\frac{Z}{n^2} &= \frac{1}{2}m_e v^2 \\
v^2 &= \frac{k^2e^4}{\hbar^2}\frac{Z}{n^2} \\
v &= \frac{ke^2}{\hbar}\frac{\sqrt{Z}}{n} = \frac{1}{4\pi\epsilon_0}\frac{2\pi e^2}{h}\frac{\sqrt{Z}}{n} = \frac{e^2}{2\epsilon_0 h}\frac{\sqrt{Z}}{n} \\
\beta = \frac{v}{c} &= \frac{e^2}{2\epsilon_0 hc}\frac{\sqrt{Z}}{n} = \frac{(1.602\times10^{-19})^2}{2\cdot 8.854\times10^{-12}\cdot 6.626\times10^{-34}\cdot 3.00\times10^8}\frac{\sqrt{1}}{[1,2,3]}
\end{aligned}$$

or 0.00763, 0.00382, and 0.00254. Because these values of β are all quite small compared to 1, we are quite justified in using nonrelativistic mechanics.

Serway problem 4.16

Answer: A Li^{2+} ion might start out life as a Li atom with $Z = 3$ protons in its nucleus and 3 electrons; take one electron away and you have singly ionized lithium or Li^{+}, and take two away and you have doubly ionized lithium or Li^{2+}. You are then back to a pretty accurate picture from the Bohr model, because you have just one electron to worry about. The energies are given by

$$E_n = -\frac{Z^2}{n^2}E_0 = -\frac{3^2}{n^2}\cdot 13.60 \text{ eV}$$

or $E_1 = -122.4$ eV, $E_2 = -30.6$ eV, $E_3 = -13.6$ eV, and $E_4 = -7.65$ eV.

Serway problem 4.19

Answer: The energy of the photon is given by the energy difference between states:

$$\Delta E = E_3 - E_2 = \frac{-E_0Z^2}{3^2} - \frac{-E_0Z^2}{2^2} = E_0Z^2\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = (13.6 \text{ eV})\cdot 1^2\cdot\frac{5}{36}$$

or 1.89 eV. The wavelength is $\lambda = hc/E = 1240/1.89 = 656$ nm and the frequency is $\nu = c/\lambda = 4.57\times10^{14}$ Hz.

Serway problem 4.24

Answer: The energy of the photon is given by

$$E = -E_0Z^2\left(\frac{1}{3^2} - \frac{1}{1^2}\right) = -(13.61 \text{ eV})(1^2)\left(\frac{1}{9} - 1\right) = \frac{8}{9}(13.61 \text{ eV}) = 12.10 \text{ eV}$$

so that its wavelength is $\lambda = hc/E = (1240 \text{ eV}\cdot\text{nm})/(12.10 \text{ eV}) = 102.5$ nm. The photon carries a momentum of $p = E/c = 12.10$ eV$/c$ or

$$p = \frac{E}{c} = \frac{(12.10 \text{ eV})\cdot(1.602\times10^{-19} \text{ J/eV})}{2.99\times10^8 \text{ m/s}} = 6.48\times10^{-27} \text{ kg}\cdot\text{m/s}.$$

The kinetic energy of the recoiling atom is (using mks units, and the mass of a proton for the hydrogen atom's mass)

$$E_k = \frac{p^2}{2m} = \frac{(6.48 \times 10^{-27})^2}{2 \cdot (1.673 \times 10^{-27} \text{ kg})} = 1.26 \times 10^{-26} \text{ J}$$

or, upon dividing by 1.602×10^{-19} J/eV, a kinetic energy of 7.87×10^{-8} eV. As you can see, it's quite safe to ignore the recoiling atom in calculating the energy of the emitted photon.

Serway problem 4.28

Answer: hromium has $Z = 24$. We first have to ask how much energy is made available by an electron dropping from the $n = 2$ state to the $n = 1$ state:

$$\Delta E = Z^2 E_0 \Big(\frac{1}{n_f^2} - \frac{1}{n_i^2}\Big) = (24)^2 \cdot 13.60 \cdot \Big(\frac{1}{1^2} - \frac{1}{2^2}\Big) = 5875 \text{ eV}.$$

(Note that we must have had a violent event, like absorption of an x-ray with an energy of at least $(24)^2 \cdot 13.60/1^2 = 7833$ eV, take place to rip out an electron from the $n = 1$ state to begin with. Note also that we are ignoring the fact that there is some screening of the nuclear charge by the *other* $n = 1$ electron; remember that we talked about $(Z - 1)^2$ in Moseley's law?) But let us plow ahead. We now have some energy we can spend; part of what we pay has to go to unbinding a $n = 4$ electron, and we can then spend what's left on kinetic energy. The binding energy of a $n = 4$ electron is

$$E_n = -\frac{Z^2}{n^2} E_0 = -\frac{24^2}{4^2} 13.6 = -490 \text{ eV}$$

(of course we're lying a bit here because there is some screening of the nuclear charge by the other electrons, but this is at least a ballpark estimate). So, we have 5875 eV to give, 490 eV of it has to go to removing the $n = 4$ electron, and 5875-490=5386 eV is left over to go into kinetic energy of the ejected electron.