PHY 251 Spring 2008: homework problem set 4, due Thursday, Feb. 28.

Serway problem 3.22

Answer: Start with

$$qvB = \gamma m \frac{v^2}{r} \text{ leading to } qBr = \gamma mv$$

which gives

$$\gamma v = \frac{qBr}{m} = \frac{1.6 \times 10^{-19} \dot{2} \times 10^{-5} \cdot 0.2}{9.11 \times 10^{-31}} = 7.03 \times 10^5 \text{ m/s}$$

Now already we can guess that we can ignore the γ in this; but let's look into it a bit more. Let's consider the difference $\gamma\beta - \beta$ in the low β limit to get a handle on this:

$$\begin{aligned} \gamma\beta - \beta &= \frac{\beta}{\sqrt{1-\beta^2}} - \beta \\ &= \beta(1 + \frac{1}{2}\beta^2) - \beta = \frac{1}{2}\beta^3 \end{aligned}$$

from which we can say that

$$\beta = \gamma\beta - \frac{1}{2}\beta^3$$

We have γv calculated above which gives

$$\gamma\beta = \gamma v/c = (7.03 \times 10^5)/(3 \times 10^8) = 2.33 \times 10^{-3}.$$

Therefore if $\gamma\beta$ is small we can say that $\gamma\beta \simeq \beta$ with an error corresponding to β^3 so our answer will be good to a few parts in a billion. Anyway, we now know the kinetic energy of the electron to be in the non-relativistic limit:

$$\frac{1}{2}mc^2\beta^2 = \frac{1}{2} \cdot (511 \times 10^3 \text{ eV}) \cdot (2.33 \times 10^{-3})^2 = 1.39 \text{ eV}$$

Then from $E_k = h\nu - \phi$ we have

$$\phi = \frac{hc}{\lambda} - 1.39 = \frac{1240}{450} - 1.39 = 1.37 \text{ eV}$$

Serway problem 3.24

Answer: The Compton relationship is

$$\begin{aligned} \Delta\lambda &= \lambda' - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta) = \frac{hc}{m_e c^2}(1 - \cos\theta) \\ &= \frac{1240 \text{ eV} \cdot \text{nm}}{511 \times 10^3 \text{ eV}}(1 - 0) = 0.00242 \text{ nm} \end{aligned}$$

and the energy imparted to the electron is equal to the energy lost by the x-ray photon or

$$\begin{aligned} \Delta E &= \frac{hc}{\lambda_0} - \frac{hc}{\lambda'} = hc(\frac{1}{\lambda_0} - \frac{1}{\lambda_0 + \Delta\lambda}) \\ &= (1240 \text{ eV} \cdot \text{nm})(\frac{1}{0.200} - \frac{1}{0.200 + 0.00242}) = 74.1 \text{ eV} \end{aligned}$$

Serway problem 3.25

Answer: The Compton wavelength shift is

$$\lambda' - \lambda_0 = \frac{hc}{mc^2}(1-\cos\theta) = \frac{1240\text{ eV}\cdot\text{nm}}{511\times 10^3\text{ eV}}(1-\cos 30^\circ) = 3.25\times 10^{-4}\text{ nm}$$

Let's get the energy of the scattered photon:

$$\begin{aligned}
\lambda' &= \lambda_0 + (\lambda' - \lambda_0) \\
\frac{hc}{\lambda'} &= \frac{hc}{\lambda_0 + (\lambda' - \lambda_0)} \\
E' &= \frac{1}{\lambda_0/hc + (\lambda'-\lambda_0)/hc} = \frac{1}{1/E_0 + (\lambda'-\lambda_0)/hc} \\
E' &= \frac{1}{1/300\times 10^3\text{ eV} + (3.25\times 10^{-4}\text{ nm})/(1240\text{ eV}\cdot\text{nm})} = 278\times 10^3\text{ eV}
\end{aligned}$$

or 278 keV. The kinetic energy of the scattered electron is then 300-278=22 keV.

Serway problem 3.28

Answer: By conservation of energy, the energy of the incident photon must be 120+40=160 keV which means its wavelength is

$$\lambda_0 = \frac{hc}{E_0} = \frac{1240\text{ eV}\cdot\text{nm}}{160\times 10^3\text{ eV}} = 7.75\times 10^{-3}\text{ nm}.$$

The angle of the Compton photon can be found from

$$\begin{aligned}
\lambda' - \lambda_0 &= \frac{h}{m_e c}(1-\cos\theta) \\
\frac{hc}{E'} - \frac{hc}{E_0} &= \frac{hc}{m_e c^2}(1-\cos\theta) \\
\frac{m_e c^2}{E'} - \frac{m_e c^2}{E_0} &= 1 - \cos\theta \\
\cos\theta &= 1 + \frac{m_e c^2}{E_0} - \frac{m_e c^2}{E'} \\
\theta &= \text{arccos}\left(1 + \frac{511}{160} - \frac{511}{120}\right) = \text{arccos}(-0.0646) = 93.7^\circ.
\end{aligned}$$

The recoil angle of the electron can be found from conservation of momentum. In the $\hat{y}$ direction, we have a net momentum of zero before and therefore after the collision, so (referring to Serway Fig. 3.24) we have

$$\begin{aligned}
p_{\lambda'}\sin\theta &= p_e \sin\phi \\
\frac{h}{\lambda'}\sin\theta &= \gamma m_e \beta c \sin\phi \\
\phi &= \text{arcsin}\left(\frac{hc}{\lambda'}\frac{1}{\gamma\beta m_e c^2}\right) = \text{arcsin}\left(\frac{E'}{\gamma\beta m_e c^2}\right).
\end{aligned}$$

Now for a 40 keV electron we have $(\gamma - 1) = 40/511$ or $\gamma = 1 + 40/511 = 1.078$ which means the approximation of $\gamma = 1 + (1/2)\beta^2$ is pretty good and $\beta = \sqrt{2(\gamma - 1)} = \sqrt{2(40/511)} = 0.396$. Therefore we have

$$\phi = \arcsin\left(\frac{120}{1.078 \cdot 0.396 \cdot 511}\right) = 33.4°.$$

Serway problem 3.30

Answer: We're told that the maximum kinetic energy given to the electron is $E_k = 30$ keV. The Compton formula maximizes the wavelength shift (and the energy transferred from the incident photon to the electron) when the term $(1 - \cos\theta)$ is maximized, or $\theta = \pi$. In this case the Compton relationship becomes

$$\lambda' - \lambda_0 = 2\frac{hc}{mc^2} \qquad \Rightarrow \qquad \lambda' = \lambda_0 + 2\frac{hc}{mc^2}$$

and at the same time the conservation of energy relationship is

$$\begin{aligned}
E_k &= \frac{hc}{\lambda_0} - \frac{hc}{\lambda'} \\
\lambda_0\lambda' E_k &= hc(\lambda' - \lambda_0) = 2\frac{(hc)^2}{mc^2} = 2\frac{h^2}{m} \\
\lambda_0\left(\lambda_0 + 2\frac{hc}{mc^2}\right)E_k &= 2\frac{h^2}{m} \\
E_k\lambda_0^2 + 2E_k\frac{h}{mc}\lambda_0 - 2\frac{h^2}{m} &= 0
\end{aligned}$$

so we have a quadratic equation of the form $ax^2 + bx + c = 0$ which has solutions of

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
\Rightarrow \qquad \lambda_0 &= \frac{-2E_k h/(mc) \pm \sqrt{4E_k^2 h^2/(mc)^2 - 4 \cdot E_k \cdot (-2)h^2/m}}{2E_k} \\
&= \frac{h}{mc}\frac{-2E_k h/(mc) \pm \sqrt{\left[2E_k h/(mc)\right]^2 + 2 \cdot \left[2E_k h/(mc)\right]^2 \cdot (mc^2/E_k)}}{2E_k h/(mc)} \\
&= \frac{hc}{mc^2}\left(-1 \pm \sqrt{1 + 2mc^2/E_k}\right).
\end{aligned}$$

Now we are interested in positive solutions for the wavelength, so we have

$$\begin{aligned}
\lambda_0 &= \frac{hc}{mc^2}\left(\sqrt{1 + 2\frac{mc^2}{E_k}} - 1\right) \\
&= \frac{1240 \text{ eV} \cdot \text{nm}}{511 \times 10^3 \text{ eV}}\left(\sqrt{1 + 2\frac{511 \times 10^3 \text{ eV}}{30 \times 10^3 \text{ eV}}} - 1\right) \\
&= 0.012 \text{ nm}
\end{aligned}$$

or $E_{\lambda_0} = hc/\lambda_0 = 104$ keV.

Serway problem 3.36

Answer: The incident photon must have an energy of $E_0 = E' + E_k = 80 + 25$ keV, so its wavelength is

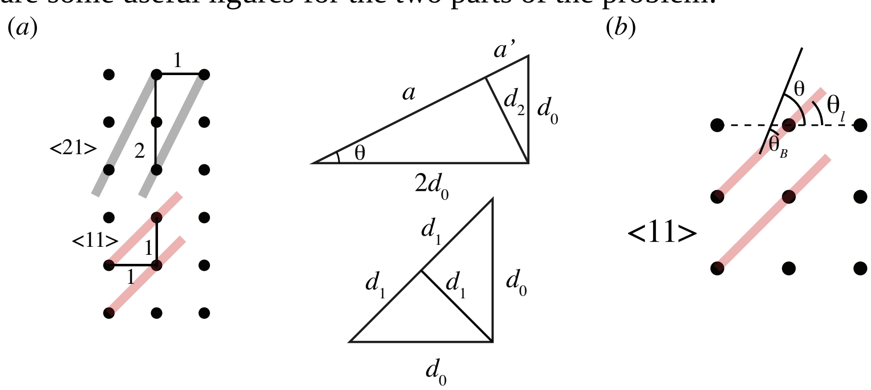
$$\lambda_0 = \frac{hc}{E_0} = \frac{1240 \text{ eV} \cdot \text{nm}}{105 \times 10^3 \text{ eV}} = 11.8 \times 10^{-3} \text{ nm}$$

We can also find $\lambda' = hc/E' = 15.5 \times 10^{-3}$ nm. We then have

$$\begin{aligned}
\lambda' - \lambda_0 &= \frac{hc}{mc^2}(1 - \cos\theta) \\
\frac{\lambda' - \lambda_0}{hc} mc^2 &= 1 - \cos\theta \\
\cos\theta &= 1 - \frac{\lambda' - \lambda_0}{hc} mc^2 \\
\theta &= \cos^{-1}\Big[1 - \frac{\lambda' - \lambda_0}{hc} mc^2\Big] \\
&= \cos^{-1}\Big[1 - \frac{15.5 \times 10^{-3} - 11.8 \times 10^{-3}}{1240} 511 \times 10^3\Big] = 121^\circ
\end{aligned}$$

Serway problem 3.38

Answer: Here are some useful figures for the two parts of the problem:



(a) Solving for d_1 is easy:

$$d_0^2 = 2d_1^2 \qquad \Rightarrow \qquad d_1 = \frac{d_0}{\sqrt{2}} = 0.707\, d_0.$$

For d_2, we have $\theta = \tan^{-1}(1/2)$ and

$$d_2 = 2d_0 \sin(\theta) = 2d_0 \sin\Big(\tan^{-1}(1/2)\Big) = 0.894\, d_0.$$

(b) There are three angles involved here: θ_l which is the angle of the lattice relative to the crystal surface ($\theta_l = 45^\circ$ for the $\langle 11 \rangle$ plane), θ_B which is the Bragg angle relative to

the lattice plane angle, and θ which is the angle of the diffracted beam relative to the crystal plane surface. Examination of the figure above leads to $\theta = \theta_B + \theta_l$, and the Bragg condition is $2d_1 \sin\theta_B = n\lambda$. Since $\theta_l = 45°$ for the $\langle 11 \rangle$ plane, we have

$$\begin{aligned} \theta &= \theta_B + \theta_l = \sin^{-1}\left(\frac{n\lambda}{2d_1}\right) + 45° = \sin^{-1}\left(\frac{n\lambda}{2d_0/\sqrt{2}}\right) + 45° \\ &= \sin^{-1}\left(\frac{[1,2,3]\cdot(0.626\times 10^{-10}\text{ m})}{1.414\cdot(4.00\times 10^{-10}\text{ m})}\right) + 45° \\ &= 51.4°,\ 57.8°,\ 64.4°. \end{aligned}$$

Serway problem 3.39

Answer: From Bragg's law we have

$$d = \frac{n\lambda}{2\sin\theta} = \frac{1\cdot 0.626\times 10^{-10}}{2\sin(6.41°)} = 2.80\times 10^{-10}\text{ meters.}$$

If you look at Serway Fig. P3.39, you see that a box of dimensions d^3 has the four times 1/8$^{\text{th}}$ the mass of a Cl^- ion and four times 1/8$^{\text{th}}$ the mass of a Na^+ ion (because only an eighth of the volume of each atom's sphere is inside the box; the rest is in adjacent boxes). Therefore the density ρ is half the mass of a Cl^- ion and half the mass of a Na^+ ion in a volume of d^3. The number of atoms per volume n (which here is eight 1/8$^{\text{th}}$s or one atom per d^3) can also be expressed in terms of the average molecular weight A and Avogadro's number N_A as

$$\begin{aligned} n\,(\text{atoms/cm}^3) &= \frac{\rho\,(\text{g/cm}^3)\cdot N_A\,(\text{atoms/mol})}{A\,(\text{g/mol})} \\ \frac{1}{d^3} &= \frac{\rho\cdot N_A}{(0.5A_{\text{Cl}} + 0.5A_{\text{Na}})} \\ N_A &= \frac{(0.5A_{\text{Cl}} + 0.5A_{\text{Na}})}{\rho\cdot d^3} \\ &= \frac{(0.5\cdot 22.99 + 0.5\cdot 35.45)\text{ g/mol}}{(2.17\text{ g/cm}^3)\cdot(2.80\times 10^{-8}\text{ cm})^3} = 6.13\times 10^{23}\text{ atoms/mol} \end{aligned}$$

which is about right!

Serway problem 3.44

Answer: If the electrons come off with a velocity of 4.2×10^5 m/s, they are clearly nonrelativistic and we can find the kinetic energy from

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mc^2\beta^2 = \frac{1}{2}(511\times 10^3\text{ eV})\cdot(\frac{4.2\times 10^5}{2.99\times 10^8})^2 = 0.50\text{ eV}.$$

We can then find the photon energy to be $h\nu = K + \varphi = 0.50 + 3.44 = 3.94$ eV. The energy per square centimeter of surface is then $(0.055\text{ W/m}^2)\cdot(10^{-2}\text{ m})^2 = 5.5\times 10^{-6}$ Joules/sec, so the photon flux is

$$\frac{(5.5\times 10^{-6}\text{ J/s})}{(3.94\text{ eV/photon})\cdot(1.602\times 10^{-19}\text{ J/eV})} = 8.71\times 10^{12}\text{ photons/sec}$$

and we can't possibly get more than one electron per photon so this is the maximum number of electrons emitted as well.

Serway problem 3.46

Answer: This is a problem where it's a bit too ugly to wait until the end to plug in numbers. We have $\lambda_0 = 0.500$ nm and $\theta = 134°$ so

$$\begin{aligned}
\lambda' - \lambda_0 &= \frac{hc}{mc^2}(1 - \cos\theta) \\
\lambda' &= \lambda_0 + \frac{hc}{mc^2}(1 - \cos\theta) \\
&= 0.500 \text{ nm} + \frac{1240 \text{ eV} \cdot \text{nm}}{511 \times 10^3 \text{ eV}}(1 - \cos 134°) = 0.5041 \text{ nm}.
\end{aligned}$$

Conservation of energy then says

$$E_k = \frac{hc}{\lambda_0} - \frac{hc}{\lambda'} = (1240 \text{ eV} \cdot \text{nm}) \left(\frac{1}{0.500 \text{ nm}} - \frac{1}{0.5041 \text{ nm}}\right) = 20.2 \text{ eV}$$

so the electron is very non-relativistic and we can use $E_k = p^2/(2m)$ or $p = \sqrt{2mE_k}$ as a way to find the electron's momentum. Our conservation of momentum relationship is that $p_{y,\lambda'} = p_{y,e}$ or if we say that the electron scatters down at an angle φ we have

$$\begin{aligned}
\frac{h}{\lambda'}\sin\theta &= \sqrt{2mE_k}\sin\varphi \\
\varphi &= \sin^{-1}\left(\frac{h\sin\theta}{\lambda'\sqrt{2mE_k}}\right) \\
&= \sin^{-1}\left(\frac{hc\sin\theta}{\lambda'\sqrt{2mc^2\,E_k}}\right) \\
&= \sin^{-1}\left(\frac{(1240 \text{ eV} \cdot \text{nm})\ \sin(134°)}{(0.5041 \text{ nm}) \cdot \sqrt{2 \cdot (511 \times 10^3 \text{ eV}) \cdot (20.2 \text{ eV})}}\right) = 23°
\end{aligned}$$