PHY 251 Spring 2008: homework problem set 3, due Thursday, Feb. 21.

Serway problem 2.21

Answer: Conservation of energy gives

$$(E_K + m_e c^2) + m_e c^2 = 2E_\gamma$$

because we go from an electron with kinetic energy E_K and the mass-energy of the electron plus the positron (same mass as electron) to two photons of identical energy E_γ. Since $m_e = 511$ keV/c^2 we have $E_\gamma = (1.000 + 0.511 + 0.511)/2 = 1.011$ MeV. Note for the electron we have

$$\begin{aligned} E_K &= (\gamma - 1)m_e c^2 \\ \gamma &= 1 + \frac{E_k}{m_e c^2} = 1 + \frac{1.000}{0.511} = 2.957 \end{aligned}$$

from which we can find $\beta = \sqrt{1 - 1/\gamma^2} = 0.941$. At the same time, conservation of momentum gives

$$\begin{aligned} p_e &= 2p_\gamma \cos\theta \\ \gamma m_e v &= 2\frac{E_\gamma}{c}\cos\theta \\ \gamma\beta m_e c^2 &= 2E_\gamma \cos\theta \\ \cos\theta &= \frac{\gamma\beta m_e c^2}{2E_\gamma} \\ \theta &= \cos^{-1}\Big(\frac{\gamma\beta m_e c^2}{2E_\gamma}\Big) = \cos^{-1}\Big(\frac{2.957 \cdot 0.941 \cdot 0.511}{2 \cdot 1.011}\Big) = \cos^{-1}(0.703) = 45.3^\circ \end{aligned}$$

Serway problem 2.23

Answer: The energy of the intial particle is $E = mc^2$ or

$$E = (3.34 \times 10^{-27}) \cdot (3 \times 10^8)^2 \cdot \frac{1 \text{ eV}}{1.602 \times 10^{-19} \text{ J}} = 1.88 \times 10^9 \text{ eV}$$

or $E = 1.88$ GeV. This goes into the total energy of two particles following the decay:

$$E = \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2$$

The two particles fly off with equal and opposite momenta. We know $\beta_1 = .987$ so $\gamma_1 = 6.22$, and $\beta_2 = .868$ so $\gamma_2 = 2.01$. We can then say

$$\begin{aligned} \gamma_1 m_1 c^2 \beta_1 / c &= \gamma_2 m_2 c^2 \beta_2 / c \\ m_1 &= m_2 \frac{\gamma_2 \beta_2}{\gamma_1 \beta_1} \end{aligned}$$

Substituting this into the expression for the total energy, we have

$$\begin{aligned} E &= \gamma_1 m_2 c^2 \frac{\gamma_2\beta_2}{\gamma_1\beta_1} + \gamma_2 m_2 c^2 \\ &= \gamma_2 m_2 c^2 \Big(1 + \frac{\beta_2}{\beta_1}\Big) \\ m_2 c^2 &= \frac{E}{\gamma_2(1+\beta_2/\beta_1)} = \frac{1.88 \text{ GeV}}{2.01 \cdot (1 + .868/.987)} = 0.50 \text{ GeV} \end{aligned}$$

Now that we know m_2 we can find m_1:

$$m_1 = m_2 \frac{\gamma_2\beta_2}{\gamma_1\beta_1} = (0.50 \text{ GeV}/c^2) \frac{2.01 \cdot 0.868}{6.22 \cdot 0.987} = 0.14 \text{ GeV}/c^2$$

That is, 500 MeV/c^2 and 140 MeV/c^2.

Serway problem 2.27

Answer: The total energy supplied is

$$(1.20 \ \frac{\text{J}}{\text{s}}) \cdot (50 \text{ min}) \cdot (60 \ \frac{\text{s}}{\text{min}}) = 3600 \text{ Joules}$$

which has a mass equivalent in mks units of

$$m = \frac{E}{c^2} = \frac{3600}{(2.99 \times 10^8)^2} = 4.03 \times 10^{-14} \text{ kg}$$

or $(4.03 \times 10^{-14})/0.025 = 1.61 \times 10^{-12}$ as a fractional mass change. This mass change is too small to be measured.

Serway problem 2.28

Answer: The total energy supplied is

$$\Big(\frac{1.2 \text{ Joules}}{\text{sec}}\Big) \cdot (50.0 \cdot 60 \text{ sec}) = 3600 \text{ Joules.}$$

The mass equivalent is found from $\Delta E = (\Delta m)c^2$ to be

$$\Delta m = \frac{\Delta E}{c^2} = \frac{3600 \text{ Joules}}{(3.00 \times 10^8 \text{ m/s})^2} = 4.00 \times 10^{-14} \text{ kg}$$

which is a mass fraction of $(4.00 \times 10^{-14} \text{ kg})/(0.025 \text{ kg}) = 1.6 \times 10^{-12}$.

Serway problem 2.31

Answer: I mislead you on this one in recitation! We'll see that there are two different Lorentz factors involved; we'll use subscripts to denote them by their velocities. Here are the steps to our solution:

1. In our frame, we saw momenta $\gamma m v$ before the collision of

$$\begin{aligned} p_1 &= \frac{m(+u)}{\sqrt{1-(+u)^2/c^2}} = \gamma_u m u \\ p_2 &= \frac{(m/3)(-u)}{\sqrt{1-(-u)^2/c^2}} = -\frac{1}{3}\gamma_u m u \end{aligned}$$

   where we realize that both momenta involve the same Lorentz factor γ_u. The net momentum in our frame before the collision is therefore

$$p_{\text{before}} = p_1 + p_2 = (+1 - \frac{1}{3})\gamma_u m u = \frac{2}{3}\gamma_u m u$$

   where of course γ_u is a simple function of u. We also see a total energy of

$$E_{\text{before}} = E_1 + E_2 = \gamma_u m c^2 + \gamma_u \frac{1}{3} m c^2 = \frac{4}{3}\gamma_u m c^2.$$

2. After the collision, in our frame we will see a momentum $\gamma_{u'} M u'$ for the combined mass M, so we have

$$p_{\text{before}} = p_{\text{after}} \qquad \Rightarrow \qquad \frac{2}{3}\gamma_u m u = \gamma_{u'} M u'.$$

   so if we can find u' we can find M.

3. The problem now reduces to finding a frame shift velocity v that puts us into the center of momentum frame before. Once in that frame, we realize that M will have zero velocity after so that frame shift velocity is precisely equal to the velocity we see for M after the collision: $v = u'$.

4. Using the momentum transform relationship $p_2 = \gamma_v(p_1 - vE_1/c^2) = 0$, we have

$$\begin{aligned} p_2 = 0 &= p_{\text{before, shifted}} = \gamma_v\left(p_{\text{before}} - vE_{\text{before}}/c^2\right) \\ &= \gamma_v\left(\frac{2}{3}\gamma_u m u - v\frac{4}{3} m c^2/c^2\right) = \gamma_v m\left(\frac{2}{3}\gamma_u u - \frac{4}{3}v\right) \\ \text{giving} \quad \frac{4}{3}v &= \frac{2}{3}\gamma_u u \\ \text{or} \quad v &= \frac{1}{2}\gamma_u u \end{aligned}$$

   or since we said $v = u'$ we have

$$u' = \frac{1}{2}\gamma_u u = \frac{1}{2}\frac{u}{\sqrt{1-u^2/c^2}}$$

   from which in turn we find

$$u'^2 = \frac{1}{4}\frac{u^2}{1-u^2/c^2}$$

and

$$\gamma_{u'} = \frac{1}{\sqrt{1 - u'^2/c^2}} = \text{grunge}$$

5. Now we return to step 2 where we found

$$\begin{aligned} \frac{2}{3}\gamma_u mu &= \gamma_{u'} Mu' = \gamma_{u'} M \frac{1}{2}\gamma_u u \\ \frac{2}{3}m &= \gamma_{u'} M \\ M &= \frac{2}{3}m\frac{1}{\gamma_{u'}} = \end{aligned}$$

Must grunge out the last bit of algebra.

Textbook gives an answer of

$$M = \frac{2m}{3}\sqrt{\frac{4 - (u^2/c^2)}{1 - (u^2/c^2)}}$$

Serway problem 2.33

Answer: The energy added is

$$\Delta E = (1.79 \times 10^{17} \text{ J/s}) \cdot (365 \cdot 24 \cdot 60 \cdot 60 \text{ s}) = 5.64 \times 10^{24} \text{ J}$$

so the mass increase Δm would be

$$\Delta m = \frac{\Delta E}{c^2} = \frac{5.64 \times 10^{24} \text{ J}}{(3.00 \times 10^8 \text{ m/s})^2} = 6.27 \times 10^7 \text{ kg}$$

or a fractional mass increase of $(6.27 \times 10^7)/(5.97 \times 10^{24}) = 1.05 \times 10^{-17}$.

Serway problem 3.6

Answer: The Planck length is

$$\Big(\frac{hG}{c^3}\Big)^{\frac{1}{2}} = \Big(\frac{(6.626 \times 10^{-34} \text{ kg} \cdot \text{m}^2/\text{s}) \cdot (6.673 \times 10^{-11} \text{m}^3/\text{kg}^2/\text{s}^2)}{(2.998 \times 10^8 \text{ m/s})^3} = 4.05 \times 10^{-35} \text{ m}$$

The Planck time works out to 1.35×10^{-43} seconds, and the Planck mass works out to 5.46×10^{-8} kg. These Planck scales turn out to provide natural scalings for theories of quantum gravity (quantum mechanics plus general relativity).

Serway problem 3.8

Answer: The energies can be found from $E = h\nu$ (using $h = 4.136 \times 10^{-15}$ eV · sec) to be

a) 2.07 eV for red visible light with a frequency of 5×10^{14} Hz.

b) 4.1×10^{-5} eV for 10 GHz which is a microwave communications frequency.

c) 1.2×10^{-7} eV for 30 MHz which is at the low end of a VHF radio frequency band used in communications.

Serway problem 3.12

Answer: The energy per photon is $E = hc/\lambda$ so the photon flux is P/E or

$$\frac{10 \text{ J/s}}{hc/\lambda} = \frac{10 \text{ J/s}}{[(1240 \text{ eV} \cdot \text{nm})/(589.3 \text{ nm})] \cdot (1.602 \times 10^{-19} \text{ J/eV})} = 2.97 \times 10^{19} \text{ photons/sec}$$

Serway problem 3.14

Answer: We have $\varphi = 2.24$ eV, and

$$K = h\nu - \varphi = \frac{hc}{\lambda} - \varphi = \frac{1240 \text{ eV} \cdot \text{nm}}{350 \text{ nm}} - 2.24 = 1.30 \text{ eV}$$

and the cutoff wavelength (where K is zero) is found from

$$\frac{hc}{\lambda} = \varphi \qquad \Rightarrow \qquad \lambda = \frac{hc}{\varphi} = \frac{1240 \text{ eV} \cdot \text{nm}}{2.24 \text{ eV}} = 554 \text{ nm}$$

Serway problem 3.15

Answer: The cutoff wavelength is when the ejected electron has zero kinetic energy, giving $hf = hc/\lambda = \varphi$ or

$$\lambda = \frac{hc}{\varphi} = \frac{1240 \text{ eV} \cdot \text{nm}}{4.2 \text{ eV}} = 295 \text{ nm.}$$

If instead we have a wavelength of 200 nm, the kinetic energy of the electron would be given by

$$E_k = hf - \varphi = \frac{hc}{\lambda} - \varphi = \frac{1240 \text{ eV} \cdot \text{nm}}{200 \text{ nm}} - (4.2 \text{ eV}) = 2.0 \text{ eV}$$

so a 2.0 Volt potential will stop a 2.0 eV electron.

Serway problem 3.20

Answer: We can write the following equations, realizing that the work function is the same in the two situations:

$$\begin{aligned} E_{k,1} &= \frac{hc}{\lambda} - \varphi \\ E_{k,2} &= 2\frac{hc}{\lambda} - \varphi. \end{aligned}$$

If we subtract the first equation from the second, we have $E_{k,2} - E_{k,1} = (hc)/\lambda$ or

$$\lambda = \frac{hc}{E_{k,2} - E_{k,1}}.$$

We can then substitute this in the first equation and solve for the work function φ:

$$\varphi = \frac{hc}{\lambda} - E_{k,1} = hc\frac{E_{k,2} - E_{k,1}}{hc} - E_{k,1} = E_{k,2} - 2E_{k,1} = 4.00 - 2 \cdot 1.00 = 2.00 \text{ eV}.$$