PHY 251 Spring 2008: homework problem set 1, due Wednesday, Feb. 6.

Some general comments: you'll see that I always work out an algebraic solution first and then plug in the numbers at the end. Also, if you put mks (meter · kilogram · second) units in, you get an answer in mks units out. Therefore I tend to be lazy about writing the units in the numerical calculations.

Problem 1: Work out the speed of the earth in its orbit from first principles. That is, start from equating gravitational force and centripetal force, and use the mass of the sun, and the mean earth-sun separation distance, to calculate the earth's speed in its orbital path.

Answer: Gravity provides the centripetal force required for circular motion:

$$\begin{aligned} G\frac{m_e m_s}{r^2} &= \frac{m_e v^2}{r} \\ v^2 &= G\frac{m_s}{r} \\ v &= \sqrt{G\frac{m_s}{r}} = \sqrt{6.673\times 10^{-11}\frac{1.988\times 10^{30}}{1.496\times 10^{11}}} = 2.98\times 10^4 \text{m/s} \end{aligned}$$

Of course, once you know r, you can also find out the speed if you know how long a year is:

$$v = \frac{2\pi r}{T} = \frac{2\pi\cdot 1.496\times 10^{11}}{1 \text{ year}}\cdot\frac{1 \text{ year}}{365.25 \text{ days}}\cdot\frac{1 \text{ day}}{24 \text{ hours}}\cdot\frac{1 \text{ hour}}{3600 \text{ sec}} = 2.98\times 10^4 \text{ m/s}$$

Problem 2: Show that if one uses the Galilean relativity transformation of (Jan. 30 lecture, Eq. 3)

$$\begin{aligned} x_2 &= x_1 - vt_1 \\ y_1 &= y_2 \\ z_1 &= z_2 \\ t_1 &= t_2 \end{aligned} \tag{1}$$

and the relationships for expansion of light spheres (Sep. 7 lecture, Eqs. 1 and 2) of

$$x_1^2 + y_1^2 + z_1^2 - c^2t_1^2 = 0 \tag{2}$$

$$x_2^2 + y_2^2 + z_2^2 - c^2t_2^2 = 0. \tag{3}$$

that you get a non-general and inconsistent-with-classical-physics result.

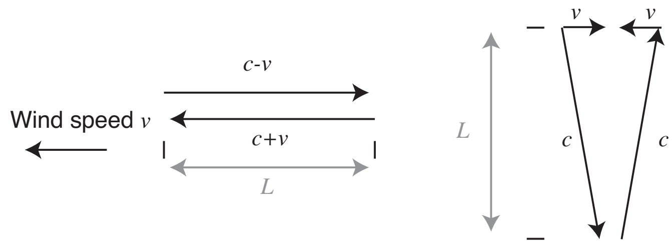
Answer: Let's do the substitutions in the second equation:

$$\begin{aligned} (x_1 - vt_1)^2 + y_1^2 + z_1^2 - c^2t_1^2 &= 0 \\ x_1^2 + y_1^2 + z_1^2 - c^2t_1^2 + v^2t_1^2 - 2vx_1t_1 &= 0 \\ \text{But} \quad x_1^2 + y_1^2 + z_1^2 - c^2t_1^2 = 0 \quad \text{so} \quad v^2t_1^2 - 2vx_1t_1 &= 0 \\ \text{giving} \quad v &= 2\frac{x_1}{t_1} \end{aligned}$$

which implies that Galilean relativity works only at a particular relationship between distance and time which differs by a factor of 2 from the usual $v = \Delta x/\Delta t$.

Serway problem 1.4

Answer: Here's a diagram:



When the airplane is flying against and with the wind, its net travel time is

$$t_1 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{L}{c}\left(\frac{1}{1-\beta} + \frac{1}{1+\beta}\right)$$

with $\beta \equiv v/c$. Now for small x the binomial expansion says $(1+x)r \simeq 1 + rx + \frac{1}{2}r(r-1)x^2 + \ldots$ so we have

$$\begin{aligned} t_1 &= \frac{L}{c}\left((1-\beta)^{-1} + (1+\beta)^{-1}\right) = \frac{L}{c}\left(1+\beta+\beta^2+\ldots+1-\beta+\beta^2+\ldots\right) \\ &\simeq 2\frac{L}{c}\left(1+\beta^2\right). \end{aligned}$$

When the plane is traveling across the wind, it has to fly at a canted direction to arrive at its destination. Its speed c' along the straight line distance from start to destination is found from $c^2 = c'^2 + v^2$ which gives

$$c' = \sqrt{c^2 - v^2} = c\sqrt{1-\beta^2}.$$

This applies the same way to both cases. The travel time is then

$$\begin{aligned} t_2 &= \frac{L}{c}\left(\frac{1}{\sqrt{1-\beta^2}} + \frac{1}{\sqrt{1-\beta^2}}\right) = 2\frac{L}{c}\left((1-\beta^2)^{-1/2}\right) \\ &= 2\frac{L}{c}\left(1 + \frac{1}{2}\beta^2 + \mathcal{O}(\beta^4)\right) \end{aligned}$$

where $\mathcal{O}(\beta^4)$ means that the next additional terms are of order β^4 which is very small if β is small. We can then subtract the two times:

$$\Delta t = t_1 - t_2 = 2\frac{L}{c}\left(1+\beta^2\right) - 2\frac{L}{c}\left(1+\frac{1}{2}\beta^2\right) = 2\frac{L}{c}\frac{1}{2}\beta^2 = \frac{L}{c}\beta^2$$

With $L = 100$ miles, $c = 500$ m/h, and $v = 100$ m/h, we have

$$\Delta t = (100/500)\cdot(100/500)^2 = (1/5)^3 = 0.008 \text{ hours}$$

or 29 seconds.

Serway problem 1.6

Answer: In $\ell' = \ell_0/\gamma$, we want $\ell'/\ell_0 = (0.5\text{ m})/(1.0\text{ m}) = 0.5 = 1/\gamma$. We then want to find β from γ:

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\beta^2}} \\
\frac{1}{\gamma^2} &= 1-\beta^2 \\
\beta^2 &= 1-\frac{1}{\gamma^2} \\
\beta &= \sqrt{1-1/\gamma^2} = \sqrt{1-1/(2)^2} = \sqrt{3/4} = \sqrt{3}/2 = 0.866
\end{aligned}$$

so we have $v = 0.866c$.

Serway problem 1.7

Answer: Again, time dilates by a factor γ or we can go to Serway Eq. 1.9 of $\Delta t = \gamma\,\Delta t_p$. In this case it is more convenient to write the proper time (time on the spacecraft) as $\Delta t_p = T$, and the time we observe which is dilated as $\Delta t = T + \Delta T$. Also note that for low velocities we can approximate the Lorentz factor γ using the binomial series expansion:

$$\gamma = (1-\beta^2)^{-1/2} \simeq 1 + \frac{1}{2}\beta^2.$$

We then have

$$\begin{aligned}
T + \Delta T &= \gamma T \\
\Delta T &= (\gamma - 1)T \\
\frac{\Delta T}{T} &= \gamma - 1 = 1 + \frac{1}{2}\beta^2 - 1 = \frac{1}{2}\beta^2 \\
\beta &= \sqrt{\frac{2\Delta T}{T}} = \sqrt{\frac{2 \cdot 1\text{ sec}}{(24\text{ h}) \cdot (3600\text{ s/h})}} = 0.0048
\end{aligned}$$

or $0.0048 \cdot 3 \times 10^8$ m/s=1.44×10^6 m/s.

Serway problem 1.10

Answer: We see a dilated time. In this case

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.95^2}} = 3.20$$

so we see a time of $t' = \gamma t_0 = 3.20 \cdot 2.6 \times 10^{-8}$ seconds or 8.32×10^{-8} seconds (we should really just say 8.3×10^{-8} seconds to reflect the fact that the knew t_0 to only two significant digits). To us, it seems to be going at $v = 0.95c$ for 8.3×10^{-8} seconds so we see it travel a distance of

$$x = vt = 0.95 \cdot (2.99 \times 10^8\text{ m/s}) \cdot (8.3 \times 10^{-8}\text{ s}) = 24\text{ meters}$$

Serway problem 1.11

Answer: The ground observer sees the jet in a frame moving at $v = 400$ m/s, which is small compared to the speed of light so we will again want to use the approximation $\gamma = 1 + \frac{1}{2}\beta^2$. The proper time of the clock in the jet is $t_0 = 3600$ s, so the time in our frame is $t' = \gamma t_0$. The additional time that we see is

$$\begin{aligned}\Delta t \equiv (t' - t_0) &= \gamma t_0 - t_0 = (\gamma - 1)t_0 \\ &\simeq (1 + \frac{1}{2}\beta^2 - 1)t_0 = \frac{1}{2}\beta^2 t_0 = \frac{1}{2}\left(\frac{400}{3\times 10^8}\right)^2 (3600\text{ s}) = 3.2\times 10^{-9}\text{ s}\end{aligned}$$

or 3.2 nanoseconds.

Serway problem 1.15

Answer: From our lecture notes, or Serway Eq. 1.15, we have

$$\nu' = \nu_0 \frac{(1-\beta)^{1/2}}{(1+\beta)^{1/2}} = \nu_0(1-\beta)^{1/2}(1+\beta)^{-1/2} \simeq \nu_0(1-\frac{1}{2}\beta)(1-\frac{1}{2}\beta) \simeq \nu_0(1-\beta)$$

We can then find

$$\frac{\Delta f}{f} = \frac{\nu' - \nu_0}{\nu_0} = \frac{\nu_0(1-\beta) - \nu_0}{\nu_0} = \frac{-\beta\nu_0}{\nu_0} = -\beta.$$

Since $\lambda = c/\nu$, we have

$$\begin{aligned}d\lambda &= d\left(\frac{c}{\nu}\right) = -c\frac{d\nu}{\nu^2} \\ \frac{\nu}{c}d\lambda &= -\frac{d\nu}{\nu} \\ \frac{d\lambda}{\lambda} &= -\frac{d\nu}{\nu}\end{aligned}$$

so

$$\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\nu}{\nu} \simeq -(-\beta) = \beta$$

Now if $\Delta\lambda = 20$ nm and $\lambda = 397$ nm, we have

$$\beta = \frac{\Delta\lambda}{\lambda} = \frac{20}{397} = 0.050$$

or $v = 0.050c = 1.5\times 10^7$ m/sec.

Serway problem 1.17

Answer: For a source moving straight towards the observer the relativistic Doppler shift is (see Serway Eq. 1.15)

$$\begin{aligned}f_{\text{obs}} &= \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_{\text{source}} \\ \frac{c}{\lambda_{\text{obs}}} &= \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}}\frac{c}{\lambda_{\text{source}}} \\ \left(\frac{\lambda_{\text{source}}}{\lambda_{\text{obs}}}\right)^2 &= \frac{1+\beta}{1-\beta}\end{aligned}$$

If we define

$$A \equiv \left(\frac{\lambda_{\text{source}}}{\lambda_{\text{obs}}}\right)^2$$

we have

$$\begin{aligned}
A &= \frac{1+\beta}{1-\beta} \\
A - A\beta &= 1+\beta \\
A - 1 &= (A+1)\beta \\
\text{giving } \beta &= \frac{A-1}{A+1} = \frac{(\lambda^2_{\text{source}} - \lambda^2_{\text{obs}})/\lambda^2_{\text{obs}}}{(\lambda^2_{\text{source}} + \lambda^2_{\text{obs}})/\lambda^2_{\text{obs}}} \\
\text{or } \beta &= \frac{\lambda^2_{\text{source}} - \lambda^2_{\text{obs}}}{\lambda^2_{\text{source}} + \lambda^2_{\text{obs}}}.
\end{aligned}$$

Now we have $\lambda_{\text{source}} = 550$ nm, so we find $\beta = 0.198$ for $\lambda_{\text{obs}} = 450$ nm, and $\beta = -0.237$ for $\lambda_{\text{obs}} = 700$ nm. Light is blue-shifted from approaching ($\beta > 0$) sources, and red-shifted from receding ($\beta < 0$) sources

Serway problem 1.18

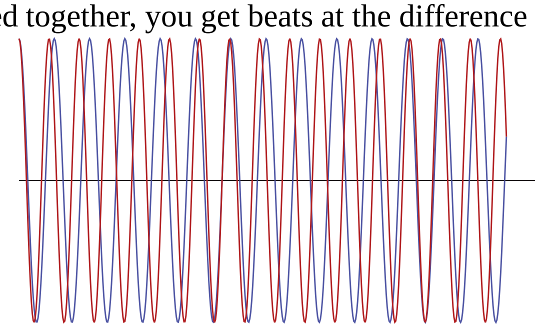
Answer: There are *two* Doppler shifts here! The radar transmitter puts out a signal at a frequency ν_1, but the car which is moving towards the transmitter ($\theta = \pi$) sees a Doppler-shifted frequency ν_2 of

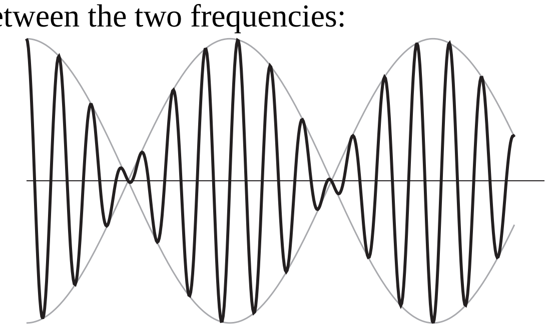
$$\nu_2 = \frac{\nu_1}{\gamma\,(1+\beta\cos(\pi))} = \nu_1\frac{\sqrt{1-\beta^2}}{1-\beta} = \nu_1\frac{\sqrt{1-\beta}\sqrt{1+\beta}}{1-\beta} = \nu_1\frac{\sqrt{1+\beta}}{\sqrt{1-\beta}}.$$

The car then reflects this signal so that it is a signal emitter at frequency ν_2. Since it's moving towards the radar gun, the radar gun sees a frequency ν_3 which is the Doppler-shifted version of ν_2 for which we have the same form of Doppler shift for $\theta = \pi$:

$$\nu_3 = \nu_2\frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} = \nu_1\frac{\sqrt{1+\beta}}{\sqrt{1-\beta}}\frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} = \nu_1\frac{1+\beta}{1-\beta} = \nu_1\frac{v+c}{v-c}$$

which is what we wanted to show. Now when two waves of slightly different frequencies are mixed together, you get beats at the difference between the two frequencies:





In this case, the beat frequency ν_b is

$$\begin{aligned}\nu_b &= \nu_3 - \nu_1 = \nu_1 \frac{1+\beta}{1-\beta} - \nu_1 = \nu_1(1+\beta)^1(1-\beta)^{-1} - \nu_1 \\ &\simeq \nu_1(1+\beta)(1+\beta) - \nu_1 \simeq \nu_1(1+2\beta) - \nu_1 = 2\beta\nu_1\end{aligned}$$

where we have assumed $\beta \ll 1$. For a car with a speed of $v = 30.0$ m/s and a microwave frequency of $\nu_1 = 10.0$ GHz, the beat frequency is

$$\nu_b = 2\beta\nu_0 = 2\frac{30.0}{2.99 \times 10^8} \cdot (10.0 \times 10^9) = 2.00 \times 10^3$$

or 2 kHz which is easily measured by cheap, low frequency electronics. If the measurement of the beat frequency is good to ± 5 Hz, then we have measured the speed to an accuracy of $(5/2000) = 0.0025$ or 0.25% and $0.0025 \cdot 30.0 = 0.075$ miles per hour. You're not likely to argue your way out of a speeding ticket by claiming that the radar gun is highly inaccurate!

Serway problem 1.20

Answer: We have two frames: our inertial frame S_1 where we see the electron moving at $v = +0.90c$, and the electron's interial frame S_2 which moves at a speed of $v = +0.90c$ with respect to us. In the electron's frame the proton has a velocity of $v_{2,x} = +0.70c$. We then need to use our velocity transform:

$$\begin{aligned}v_{1,x} &= \frac{v_{2,x} + v}{1 + vv_{2,x}/c^2} \text{ (Lecture 2 Eq. 13)} \qquad u_x = \frac{u'_x + v}{1 + u'_x v/c^2} \text{ (Serway Eq. 1.30)} \\ v_{1,x} &= \frac{0.70c + 0.90c}{1 + (0.70c)(0.90c)/c^2} = \frac{1.6c}{1 + .63} = 0.98c\end{aligned}$$

Serway problem 1.21

Answer: From our frame, we have spacecraft A moving at $v_A = 0.50c = v_{1,x}$, and spacecraft B moving at $v_B = 0.80c$. If we make a velocity shift $v = v_B$ to go into spacecraft B's frame, the velocity $v_{2,x}$ that spacecraft B sees spacecraft A traveling at is given by

$$v_{2,x} = \frac{v_{1,x} - v}{1 - vv_{1,x}/c^2} = \frac{0.50c - 0.80c}{1 - (0.80c)(0.50c)/c^2} = \frac{-0.30c}{1 - 0.4} = -0.5c.$$