Physics 251 final exam, Dec. 21, 2006. Here are some isotopic masses you might find to be of use:

${}^{4}_{2}$He:	4.002 603	${}^{142}_{54}$Xe:	141.929 701	${}^{143}_{55}$Cs:	142.927 330	${}^{144}_{55}$Cs:	143.932 030
${}^{145}_{55}$Cs:	144.935 390	${}^{143}_{56}$Ba:	142.920 617	${}^{144}_{56}$Ba:	143.922 940	${}^{145}_{56}$Ba:	144.926 920
${}^{143}_{57}$La:	142.916 059	${}^{144}_{57}$La:	143.919 590	${}^{145}_{57}$La:	144.921 641	${}^{235}_{92}$U:	235.043 923
${}^{236}_{92}$U:	236.045 562	${}^{237}_{92}$U:	237.048 724	${}^{238}_{92}$U:	238.050 783	${}^{239}_{92}$U:	239.054 288
${}^{237}_{93}$Np:	237.048 167	${}^{238}_{93}$Np:	238.050 941	${}^{239}_{93}$Np:	239.052 931	${}^{239}_{94}$Pu:	239.052 157
${}^{238}_{95}$Am:	235.048 032	${}^{239}_{95}$Am:	239.053 018	${}^{240}_{95}$Am:	240.055 288	${}^{239}_{96}$Cm:	239.054 951
${}^{240}_{96}$Cm:	240.055 519	${}^{241}_{96}$Cm:	241.057 647				

1. Light from the $3p \rightarrow 2d$ transition in hydrogen in atoms in a distant galaxy is observed at a wavelength of $\lambda = 850$ nm. What's the velocity of the galaxy relative to earth?
   Answer: The photon energy of the transition is given by

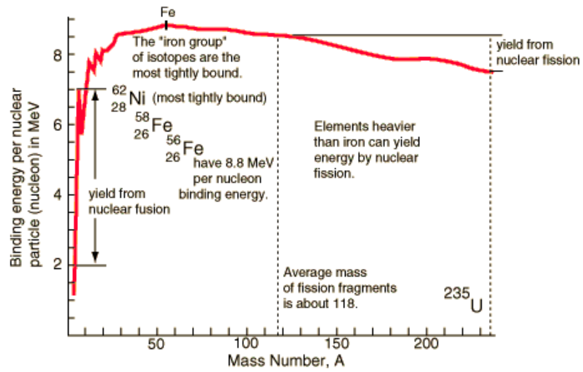
$$\Delta E = E_3 - E_2 = (-13.6\frac{1^2}{3^2}) - (-13.6\frac{1^2}{2^2}) = 1.89 \text{ eV}$$

   so that its wavelength is $\lambda_0 = hc/E = (1240/1.89) = 656$ nm. If it's instead observed at a wavelength of $\lambda = 850$ nm, we have $\nu/\nu_0 = \lambda_0/\lambda = 656/850$. The relativistic Doppler shift for a receding source involves $\theta = 0$ in

$$\begin{aligned}
\nu &= \nu_0 \frac{1}{\gamma[1+\beta\cos\theta]} = \nu_0 \frac{1}{\gamma[1+\beta]} = \nu_0 \frac{\sqrt{1-\beta^2}}{1+\beta} \\
&= \nu_0 \frac{\sqrt{(1-\beta)(1+\beta)}}{1+\beta} = \nu_0 \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \\
(\frac{\nu}{\nu_0})^2 &= \frac{1-\beta}{1+\beta} \\
\beta\left((\frac{\nu}{\nu_0})^2 + 1\right) &= 1 - (\frac{\nu}{\nu_0})^2 \\
\beta &= \frac{1-(\nu/\nu_0)^2}{1+(\nu/\nu_0)^2} = \frac{1-(656/850)^2}{1+(656/850)^2}
\end{aligned}$$

   or $\beta = 0.253$, giving $v = \beta c = (0.253) \cdot 2.99 \times 10^8$ m/sec or 7.65×10^7 m/sec.

2. Sketch the curve of nuclear binding energy per nucleon given from the liquid drop model, giving an accurate scale of MeV/nucleon. Indicate what element has the strongest binding energy per nucleon. Name the process by which heavier elements approach the element of strongest binding energy per nucleon. Name the process by which lighter elements approach the element of strongest binding energy per nucleon. Your entire answer should fit on one blue book page.
   Answer: Here's the relevant curve shown "upside down" (that is, the binding energies are negative):



3. Put the following discoveries in chronological order, and match the most relevant formula and the discoverer's name with each discovery.

Matter waves	$N(\theta) = \frac{nt}{4r^2}(\frac{zZ}{2K})^2(\frac{e^2}{4\pi\epsilon_0})^2\frac{1}{\sin^4(\theta/2)}$	Rutherford
Wave equation	$E = h\nu$	Bohr
Blackbody distribution	$L = n\hbar$	Schrödinger
Special relativity	$\gamma = 1/\sqrt{1-(v/c)^2}$	de Broglie
Half-integer-spin statistics	$E = h\nu - \varphi$	Planck
Small, dense nucleus	$\lambda = h/p$	Einstein
Photoelectric effect	$f(E) = \frac{1}{\exp[(E-E_F)/k_BT]+1}$	Fermi and Dirac
Quantized atomic orbitals	$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$	Einstein

Answer:

Blackbody distribution	$E = h\nu$	Planck
Photoelectric effect	$E = h\nu - \varphi$	Einstein
Special relativity	$\gamma = 1/\sqrt{1-(v/c)^2}$	Einstein
Small, dense nucleus	$N(\theta) = \frac{nt}{4r^2}(\frac{zZ}{2K})^2(\frac{e^2}{4\pi\epsilon_0})^2\frac{1}{\sin^4(\theta/2)}$	Rutherford
Quantized atomic orbitals	$L = n\hbar$	Bohr
Matter waves	$\lambda = h/p$	de Broglie
Wave equation	$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$	Schrödinger
Half-integer-spin statistics	$f(E) = \frac{1}{\exp[(E-E_F)/k_BT]+1}$	Fermi and Dirac

4. A ${}^{144}_{56}$Ba nucleus undergoes β^- decay. What is the maximum kinetic energy and the velocity of the emitted electron? (Assume that the neutrino carries negligible kinetic energy).
Answer: First of all, the mass of an electron in atomic mass units is

$$510.998 \times 10^3 \text{ eV}/c^2 \cdot \frac{1 \text{ u}}{931.494 \times 10^6 \text{ eV}/c^2} = 0.000\,549 \text{ u}.$$

Next, the decay we're considering is ${}^{144}_{56}\text{Ba} \rightarrow {}^{144}_{57}\text{ La} + {}^{0}_{-1}\beta^- + \nu$ which has a mass change of

$$(\text{mass before} - \text{mass after}) = (143.922\,940 - 143.919\,590 - 0.000\,549) = 0.002\,801 \text{ amu}$$

which we can express in MeV as

$$(0.002\,801 \text{ amu}) \cdot \frac{931.494 \text{ MeV}/c^2}{\text{amu}} \cdot c^2 = 2.61 \text{ MeV}$$

We then go from an atom at rest to an atom and electron flying away from each other. Because the atom is massive and the electron is very light, the electron will carry nearly all of the kinetic energy so its maximum energy will be about 2.61 MeV/c^2. Its velocity can be found from KE=$(\gamma - 1)mc^2$ giving

$$\gamma = 1 + \frac{\text{KE}}{mc^2} = 1 + \frac{2.61 \times 10^6 \text{ eV}}{510.998 \times 10^3 \text{ eV}} = 6.11$$

so the electron is quite relativistic. We can find the velocity from

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} \\ \frac{1}{\gamma^2} &= 1 - \beta^2 \\ \beta &= \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 1/6.11^2} = 0.987 \end{aligned}$$

so $v = 0.987c$.

5. Sodium is a monovalent metal having a density of 0.971 g/cm^3 and a molar mass of 23.0 g/mol. Use this information to calculate A) the density of charge carriers, B) the Fermi energy, and C) the Fermi speed for sodium.
Answer: For sodium, the atom number density is

$$\frac{N}{V} = \frac{\rho\, N_A}{A} = \frac{(0.971 \text{ g/cm}^3) \cdot (6.02 \times 10^{23} \text{ atoms/mol})}{23.0 \text{ g/mol}} = 2.54 \times 10^{22} \text{ atoms/cm}^3$$

and since it is monovalent rather than di- or trivalent there is one valence electron per atom. The Fermi energy is then

$$\begin{aligned} E_F &= \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3} = \frac{(hc)^2}{8mc^2}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3} \\ &= \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8 \cdot (511 \times 10^3 \text{ eV})}\left(\frac{10^{-7} \text{ cm}}{1 \text{ nm}}\right)^2 \left(\frac{3}{\pi}\,(2.54 \times 10^{22}\,\frac{1}{\text{cm}^3})\right)^{2/3} = 3.15 \text{ eV} \end{aligned}$$

The Fermi speed is found from $(1/2)mv^2 = E_F$ which gives a Fermi speed of

$$v_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2 \cdot (3.15 \text{ eV})}{511 \times 10^3 \text{ eV/c}^2}} = 0.00351c = 1.05 \times 10^6 \text{ m/s.}$$

6. ^{239}Pu alpha decays with a half-life of 2.41×10^4 years. Calculate the energy of the alpha particle, or if you can't, assume that it is 10 MeV. Compute the power output, in watts, which could be obtained (at 100% efficiency) from 1.0 kg of "fresh" ^{239}Pu. *Thermal power sources like this have been used to power some craft launched to deep space where sunlight is weak; a recent example is the Cassini probe to Saturn (the team is led by a woman who earned her B.A. in Astronomy at Stony Brook).*
Answer:

The reaction is $^{239}_{94}\text{Pu} \rightarrow^4_2 \alpha +^{235}_{92} \text{U}$. The energy released per decay is

$$Q = m_{\text{Pu}} - m_\alpha - m_{\text{U}} = 239.052\,157 - 4.002\,603 - 235.043\,923 = 0.005\,631 \text{ u}$$

or $0.00563 \cdot 931.494 = 5.29$ MeV. Now we will want to know the number of seconds per year:

$$(60 \text{ s/m}) \cdot (60 \text{ m/h}) \cdot (24 \text{ h/d}) \cdot (365 \text{ d/y}) = 3.15 \times 10^7 \text{ s/y}.$$

The activity is given by $R = \lambda N$ or, since $t_{1/2} = \ln 2/\lambda$, the activity is

$$\begin{aligned} R &= \lambda N = \frac{\ln 2}{t_{1/2}} \cdot \frac{m\,N_A}{A} \\ &= \frac{\log 2}{2.41 \times 10^4 \text{ y}} \cdot \frac{1 \text{ y}}{3.15 \times 10^7 \text{ s}} \cdot \frac{1 \times 10^3 \text{ g}}{239.052 \text{ g/mole}} \cdot \frac{6.02 \times 10^{23} \text{ atoms}}{\text{mole}} \\ &= 2.30 \times 10^{12} \frac{\text{decays}}{\text{sec}} \end{aligned}$$

The power is then

$$\frac{5.29 \times 10^6 \text{ eV}}{\text{decay}} \cdot 2.30 \times 10^{12} \frac{\text{decays}}{\text{sec}} \cdot 1.602 \times 10^{-19} \frac{\text{J}}{\text{eV}} = 1.95 \text{ W}$$

7. Let's assume that you are at Times Square on New Year's Eve and a terrorist releases some finely ground powder of ^{239}Pu into the air. You inhale one 0.1 μg particle which becomes permanently lodged in your lungs. Let's assume that the alpha particles created by its decay deposit 50% of their energy within a distance of 10^{-2} cm in tissue. Assume that the tissue has the density of water.
A) (8 points) What is the radiation dose in Gray received by this tissue over the course of a week?
B) (2 points; if you don't have an answer to part A, assume the result is 10^4 Gray). What can you say about how large a dose this is?
Answer: This isn't needed for solving the problem, but a mass of 1.0 μg of Pu can be contained within a sphere of diameter r given by

$$\rho = \frac{m}{V} = \frac{m}{(4/3)\pi r^3} \qquad \text{or} \qquad r = \left(\frac{3m}{4\pi\rho}\right)^{1/3} = \left(\frac{3 \cdot 1.0 \times 10^{-7} \text{ g}}{4\pi \cdot 19.8 \text{ g/cm}^3}\right)^{1/3} = 1.06 \times 10^{-3} \text{ cm}$$

or 11 μm. A particle of this size (about the size of a cell) is easily ingested or inhaled. Anyway, to solve the problem and estimate the dose D you need to multiply the activity R by the energy per decay and the duration of 1 year, and divide by the mass of tissue. Let's start by getting the activity R in decays per year:

$$\begin{aligned} R &= \lambda N = \frac{\log 2}{t_{1/2}} \cdot \frac{m\, N_A}{A} \\ &= \frac{\log 2}{2.41\times 10^4 \text{ y}} \cdot \frac{1\times 10^{-7} \text{ g}}{239.052 \text{ g/mole}} \cdot \frac{6.02\times 10^{23} \text{ atoms}}{\text{mole}} \\ &= 7.2\times 10^9 \,\frac{\text{decaying atoms}}{\text{year}} \end{aligned}$$

We can now calculate the dose (ionizing energy per mass) deposited in the tissue over a week or 1/52 of a year, assuming half is deposited in a sphere of radius 10^{-2} cm:

$$\begin{aligned} D &= \frac{R\cdot t\cdot E}{\rho\cdot (4/3)\pi r^3} \\ &= \frac{(7.2\times 10^9 \text{ decays/y})\cdot(1/52 \text{ y})\cdot(0.5)\cdot(5.29\times 10^6 \text{ eV/decay})\cdot(1.602\times 10^{-19} \text{ J/eV})}{(1 \text{ g/cm}^3)\cdot(4/3)\pi(10^{-2} \text{ cm})^3\cdot(10^{-3} \text{ kg/g})} \\ &= 1.4\times 10^4 \text{ J/kg} \end{aligned}$$

or 14,000 Grays. Since $\text{LD}_{50} \simeq 5$ Gray, this is a very high local dose and your goose is cooked. (Actually, LD_{50} refers to a whole body dose and your gastrointenstinal tract and central nervous system show the most sensitivity to radiation dose; the likely effect of the Pu particle is to cause lung cancer through this very high localized radiation dose).

8. In a piece of rock from the Moon, the ^{87}Rb content is found to be 1.82×10^{10} atoms per gram of material and the ^{87}Sr content is found to be 1.07×10^9 atoms per gram. The relevant decay is $^{87}\text{Rb} \rightarrow {}^{87}\text{Sr} + \beta^-$ with a half-life of 4.8×10^{10} years. Determine the age of the rock, assuming that there was no ^{87}Sr present when the rock was formed.
Answer: In one gram of material we have $A = 1.82\times 10^{10}$ atoms of ^{87}Rb, and $B = 0.107\times 10^{10}$ atoms of ^{87}Sr. If we started out with all ^{87}Rb and no ^{87}Sr, we could say that $(A+B)$ represents N_0, the initial number of ^{87}Rb atoms, and A represents the number left after decay. We would then have

$$\begin{aligned} N &= N_0 \exp[-\lambda t] = N_0 \exp[-\ln(2)\,\frac{t}{t_{1/2}}] \\ A &= (A+B)\exp[-\ln(2)\,\frac{t}{t_{1/2}}] \\ \ln\left[\frac{A}{A+B}\right] &= -\ln(2)\,\frac{t}{t_{1/2}} \\ \ln\left[\frac{A+B}{A}\right] &= \ln(2)\,\frac{t}{t_{1/2}} \\ t &= \frac{t_{1/2}}{\ln(2)}\ln\left[\frac{A+B}{A}\right] = \frac{4.8\times 10^{10} \text{ years}}{\ln(2)}\ln\left[\frac{(1.82+0.107)\times 10^{10}}{1.82\times 10^{10}}\right] = 3.96\times 10^9 \text{ years} \end{aligned}$$

or 3.96 billion years old. If in fact the "fresh" rock already had some ^{87}Sr in it, then the initial number of atoms of ^{87}Rb would be $(A + xB)$ with $x < 1$ and we'd get an even older age for the rock.

9. Choose one of the following questions to answer. Your answer must be contained within two facing blue book pages. You don't have to use complete sentences, but you do have to convey a logical ordering to your answer; that is, a clear and shorter answer is preferred to two pages of scribbling everything you can think of which may or may not be relevant. Diagrams and relevant equations are encouraged.

   - How does a laser work?
   - How does a nuclear fission reactor work?
   - How does a diode work?
   - How does a fission bomb work?

   Answer:

   - Laser: should mention population inversion, stimulated and spontaneous emission versus absorption, Einstein A and B coefficients, optical cavity, pumping...
   - Fission reactor: should mention isotope enrichment of ^{235}U (from 0.7% to 2.5–3.5%), neutron-induced fission releases 2.5 neutrons plus 210 MeV, delayed neutrons make controllability possible, critical mass, water or graphite moderators, heat transfer to drive steam turbine...
   - Diode: should mention location of Fermi energy in a band gap, dopants produce donor states just below conduction band (n type) and acceptor states just above valence band (p type), at n-p junction one gets a voltage bias through charge transfer to get the Fermi energies to match, net result is strong conduction for forward bias of 0.6V and near-zero conduction when reverse biased.
   - High degree of isotope enrichment (70% in ^{235}U?), fissionable isotope (usually ^{235}U or ^{239}Pu), one neutron-induced fission releases about 210 MeV plus 2.5 neutrons to trigger chain reaction, rapid assembly of critical mass, huge energy release from a compact package.