

Physics 251 exam 2, March 28, 2007. You may use a calculator, and the equation sheet that I have provided.

1. (10 points) Write out the electronic configuration for neon ( $Z = 10$ ). Write out the values for the set of quantum numbers  $n$ ,  $\ell$ ,  $m_\ell$ , and  $m_s$  for each of the electrons.

*Answer:* The energy order of states is  $1s$  then  $2s$  then  $2p$ , and the electron configurations of a neon atom in its ground state are

$n$	$\ell$	$m_\ell$	$m_s$
1	0	0	$+\frac{1}{2}$
1	0	0	$-\frac{1}{2}$
2	0	0	$+\frac{1}{2}$
2	0	0	$-\frac{1}{2}$
2	1	1	$+\frac{1}{2}$
2	1	1	$-\frac{1}{2}$
2	1	0	$+\frac{1}{2}$
2	1	0	$-\frac{1}{2}$
2	1	-1	$+\frac{1}{2}$
2	1	-1	$-\frac{1}{2}$

2. (10 points) The first excited state (not the ground state) of a harmonically oscillating electron is 3.0 eV. What's the energy of the ground state? What's the oscillation frequency of the ground state in Hertz? What's the photon wavelength associated with the transition between ground and excited states?

*Answer:* Harmonic oscillator state energies go like  $(n + \frac{1}{2})\hbar\omega$  or  $\frac{1}{2}\hbar\omega$ ,  $\frac{3}{2}\hbar\omega$ , and so on. If the first excited state has an energy of 3.0 eV then we have

$$\frac{3}{2}\hbar\omega = 3.0 \text{ eV} \quad \Rightarrow \quad \frac{1}{2}\hbar\omega = 1.0 \text{ eV}$$

for the energy of the ground state. The oscillation frequency of the ground state is found from

$$E = \frac{1}{2}\hbar\omega = \frac{1}{2} \frac{h}{2\pi} 2\pi f \quad \Rightarrow \quad f = \frac{2E}{h} = \frac{2 \cdot 1.0 \text{ eV}}{4.14 \times 10^{-15} \text{ eV} \cdot \text{s}} = 4.8 \times 10^{14} \text{ Hz.}$$

The energy difference between the two states is  $3.0 - 1.0 = 2.0$  eV, so the photon wavelength associated with the transition is

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{2.0 \text{ eV}} = 620 \text{ nm.}$$

3. (10 points) Consider an atom made from a proton and a negative muon (207 times the mass of an electron). What's the energy and photon wavelength associated with a  $2p \rightarrow 1s$  transition?

*Answer:* This is almost the same as problem 4.34 which was assigned as homework. From the Bohr model we have  $E_n = -E_0 Z^2/n^2$  so the energy difference is

$$\Delta E = E_2 - E_1 = -E_0 1^2 \left( \frac{1}{2^2} - \frac{1}{1^2} \right) = \frac{3}{4} E_0.$$

However, because  $E_0 = me^4/(8\epsilon_0^2 h^2)$  is linear with reduced mass  $m$ , we need to adjust  $E_0$  to use the reduced mass of the proton plus muon, relative to just the electron mass from which  $E_0$  is defined. The reduced mass is (in units of MeV/ $c^2$ )

$$m_r = \frac{m_p \cdot 207m_e}{m_p + 207m_e} = \frac{939 \cdot (207 \cdot 0.511)}{939 + 207 \cdot 0.511} = 95.1 \text{ MeV}/c^2$$

so we must adjust  $E_0$  to be  $E'_0$  by

$$E'_0 \rightarrow E_0 \frac{m_r}{m_e} = (13.6 \text{ eV}) \frac{95.1}{0.511} = 2530 \text{ eV}$$

so the energy of the transition is  $(3/4)E'_0 = 0.75 \cdot 2530 = 1900 \text{ eV}$  and the photon wavelength is  $\lambda = hc/E = 1240/1900 = 0.653 \text{ nm}$ .

4. (10 points) Show that the wavefunction  $\psi(x) = Ce^{-ax^2}$  satisfies the Schrödinger equation for a harmonic oscillator potential  $U = \frac{1}{2}m\omega^2 x^2$ . Find  $a$ , and find the energy of the state. You have to derive these things with an explanation of your derivation; don't just write down the final answer and say "because the equation sheet says so."

*Answer:* To pop this into the Schrödinger equation, we'll need to know the second derivative:

$$\begin{aligned} \frac{d}{dx} \frac{d}{dx} Ce^{-ax^2} &= C \frac{d}{dx} (-2xa) e^{-ax^2} = C (-2ae^{-ax^2} + 4x^2 a^2 e^{-ax^2}) \\ &= (-2a + 4x^2 a^2) Ce^{-ax^2} = (4a^2 x^2 - 2a) \psi(x). \end{aligned}$$

Now we pop this into the Schrödinger equation:

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi + U\psi &= E\psi \\ -\frac{\hbar^2}{2m} (4a^2 x^2 - 2a) \psi + \frac{1}{2} m \omega^2 x^2 \psi &= E\psi \\ \frac{a\hbar^2}{m} - \frac{2a^2 \hbar^2}{m} x^2 + \frac{1}{2} m \omega^2 x^2 &= E \\ E &= \frac{a\hbar^2}{m} + \left( \frac{1}{2} m \omega^2 - \frac{2a^2 \hbar^2}{m} \right) x^2. \quad (1) \end{aligned}$$

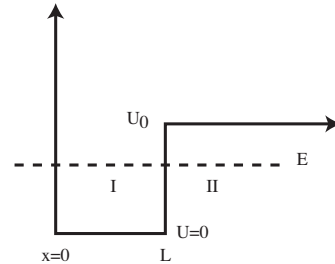
The energy of the quantum state should not depend on  $x$  so we require the term in parenthesis to be zero which gives

$$\begin{aligned} \frac{1}{2} m \omega^2 &= \frac{2a^2 \hbar^2}{m} \\ a^2 &= \frac{m^2 \omega^2}{4\hbar^2} \\ a &= \frac{m\omega}{2\hbar} \end{aligned}$$

and then from Eq. 1 we have that the energy of the state is

$$E = a \frac{\hbar^2}{m} = \frac{m\omega}{2\hbar} \frac{\hbar^2}{m} = \frac{1}{2}\hbar\omega.$$

5. (15 points) Consider the potential shown at right. Write an expression for the wave function in region I, and in region II, giving expressions for what the things are that multiply  $x$  in  $\psi(x)$  in terms of  $m$ ,  $E$ , and  $U_0$ . If  $U_0 = 5$  eV and we were to find an energy solution with  $E = 3$  eV, what's the characteristic tunneling distance of an electron in region II? Use boundary conditions to show and explain that there will be discrete energy states.



*Answer:* This is pretty much the same as problem 6.23. Because the potential goes to  $\infty$  at  $x = 0$ , we need to have  $\psi(x = 0) = 0$  so that we don't have a divergence at  $U(x \leq 0) \cdot \psi(x \leq 0)$ . In region I ( $0 \leq x \leq L$ ) we have  $U = 0$  so a valid wavefunction is (see p. 201)

$$\psi_I = A \sin(kx) \quad \text{with} \quad k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{\sqrt{2mE}}{\hbar}.$$

In region II ( $x > L$ ) we are in a classically disallowed region so we will have an exponentially decaying wave function of the form (see Serway Eq. 6.21)

$$\psi_{II} = B e^{-\alpha(x-L)} \quad \text{with} \quad \alpha = \frac{\sqrt{2m(U_0 - E)}}{\hbar}.$$

If  $U_0 = 5$  eV and we were to find that  $E = 3$  eV was a valid solution, the characteristic tunnelling distance for the probability of the particle or  $\psi^2 \propto (e^{-\alpha x})^2 = e^{-2\alpha x}$  would be

$$\begin{aligned} \frac{1}{2\alpha} &= \frac{\hbar}{2\sqrt{2m(U_0 - E)}} = \frac{hc}{8\pi\sqrt{2mc^2(U_0 - E)}} \\ &= \frac{1240 \text{ eV} \cdot \text{nm}}{8\pi\sqrt{2} \cdot (511 \times 10^3 \text{ eV}) \cdot (5 - 3 \text{ eV})} = 0.035 \text{ nm}. \end{aligned}$$

Our boundary conditions are

$$\psi_I(x = L) = \psi_{II}(x = L) \quad \text{and} \quad \left. \frac{d\psi_I}{dx} \right|_{x=L} = \left. \frac{d\psi_{II}}{dx} \right|_{x=L}$$

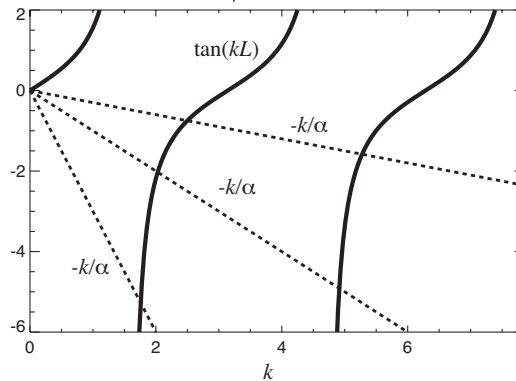
or

$$A \sin(kL) = B e^{-\alpha \cdot 0} = B \quad \text{and} \quad Ak \cos(kL) = -\alpha B e^{-\alpha \cdot 0} = -\alpha B$$

and if we divide these two equations we have

$$\frac{A \sin(kl)}{Ak \cos(kl)} = \frac{B}{-\alpha B} \quad \Rightarrow \quad \tan(kL) = -\frac{k}{\alpha}.$$

This does not give us a simple solution, but we can learn something from a graphical solution of how  $\tan(kL)$  scales with  $k$ , versus how  $-k/\alpha$  scales with  $k$ :



As you can see, no matter what slope  $-1/\alpha$  we pick, we're going to have only discrete points where  $\tan(kL)$  intersects with  $-k/\alpha$ . Once we identify a point  $k$  where such a solution occurs, we can find  $E$  from  $k = \sqrt{2mE}/\hbar$  or better yet we can write  $\tan(kL) = -k/\alpha$  as

$$\tan\left(\frac{L\sqrt{2mE}}{\hbar}\right) = -\sqrt{\frac{E}{U-E}}$$