A reminder: elements of the calculation

- We're interested in how hot a red-hot or a white-hot oven is.
- The calculation has pushed us into a first look at statistical mechanics. We will want to know of two things:
  - → *Density of available states* $g(E)$, which in this case will deal with the number of possible configurations of photons.
  - → *Probability of occupying available states* $f(E)$, which in this case will deal with how many photons are likely to be in each of various states.
- We found a volume-normalized density of available states $\rho(\nu)$ (Planck, 1897) of

$$\rho(\nu) = \frac{8\pi}{c^3}\nu^2\, d\nu. \qquad (1)$$

based on requiring that there be an integer number of half-waves inside the blackbody cavity.

Occupying available states I

- It may be foolish, but we're now going to dive into a cartoon version of statistical mechanics and thermodynamics. Don't worry too much about following every detail of the calculation; we'll revisit this a bit more later on in the course, and you'll really learn it in PHY 306, CHE 301, or MEC 301 if you take one of those courses.
- Notation used here is of **Thermal Physics** by Charles Kittel and Herbert Kroemer (W. H. Freeman and Company, 1980).
- A system $\mathcal{S}$ has $g_{\mathcal{R}}(E)$ states accessible for total energy E.
- Put this system $\mathcal{S}$ into thermal contact with a reservoir $\mathcal{R}$ which originally had a total energy U_0. The reservoir is so large that the system $\mathcal{S}$ has a weak effect on the reservoir $\mathcal{R}$.
- Think about all the half-waves of light in a box; big numbers, so let's work with the logarithm $g_{\mathcal{R}}(E)$, or

  $$\sigma_{\mathcal{R}}(E) = \log g_{\mathcal{R}}(E), \tag{2}$$

  The logarithm of the number of available states is known by a particular name in statistical mechanics: it is the *entropy* of a system.

Occupying available states II

- System $\mathcal{S}$ is in a state 1 with energy ϵ_1, or a state 2 with energy ϵ_2. What happens to the reservoir $\mathcal{R}$ as a consequence of these two choices?
- Fundamental assumption: equal likelihood for all available energy=U_0 states.
- Therefore probability P that the reservoir is in state 1 versus state 2 is simply given by the ratio of states $g_{\mathcal{R}}(E)$ accessible to the reservoir at the two energies, or

$$\begin{aligned}
\frac{P(\epsilon_1)}{P(\epsilon_2)} &= \frac{g_{\mathcal{R}}(U_0-\epsilon_1)}{g_{\mathcal{R}}(U_0-\epsilon_2)} \\
&= \frac{\exp\left[\sigma_{\mathcal{R}}(U_0-\epsilon_1)\right]}{\exp\left[\sigma_{\mathcal{R}}(U_0-\epsilon_2)\right]} \\
&= \exp\left[\sigma_{\mathcal{R}}(U_0-\epsilon_1)-\sigma_{\mathcal{R}}(U_0-\epsilon_2)\right] \qquad (3)
\end{aligned}$$

using logarithm of the density of available states, or entropy, of Eq. 2.

Occupying available states III

- Approximate the entropy with the first two terms of a Taylor expansion:

(4) $$\sigma(U_0 - \epsilon) \simeq \sigma(U_0) - \epsilon \frac{\partial \sigma}{\partial U} + \frac{\epsilon^2}{2!} \frac{\partial^2 \sigma}{\partial U^2} + \dots$$

- The quantity $(\partial\sigma/\partial U)$ measures how entropy (*i.e.*, number of states) increases as energy is added into the system.
- Define temperature τ of a system with a fixed number of particles N as

(5) $$\frac{1}{\tau} \equiv \left(\frac{\partial \sigma}{\partial U} \right)_N .$$

- Use a scale factor to relate to the usual Kelvin temperature T:

(6) $$\tau = k_B T,$$

where $k_B = 1.381 \times 10^{-23}$ Joules/Kelvin is known as Boltzmann's constant. Note: $k_B \cdot (300\ \mathrm{K}) = 0.026$ eV, or about 1/40 eV.

Occupying available states IV

- Use Taylor expansion of Eq. 4, Eq. 5 of $1/\tau \equiv (\partial\sigma/\partial U)_N$ and Eq. 6 of $\tau = k_B T$ into Eq. 3 to obtain

$$\begin{aligned}\frac{P(\epsilon_1)}{P(\epsilon_2)} &= \exp\Big[\sigma_{\mathcal{R}}(U_0-\epsilon_1) - \sigma_{\mathcal{R}}(U_0-\epsilon_2)\Big] \\ &\simeq \exp\left[\left(\sigma(U_0) - \epsilon_1\frac{\partial\sigma}{\partial U}\right) - \left(\sigma(U_0) - \epsilon_2\frac{\partial\sigma}{\partial U}\right)\right] \\ &\simeq \exp\left[-\epsilon_1\frac{1}{\tau} + \epsilon_2\frac{1}{\tau}\right] \simeq \frac{\exp\left[-\frac{\epsilon_1}{k_BT}\right]}{\exp\left[-\frac{\epsilon_2}{k_BT}\right]}. \qquad (7)\end{aligned}$$

- In other words, the *relative* likelihood of a system with temperature T choosing one particular state with energy E is given by

$$\exp[-E/k_BT] \qquad (8)$$

which is known as the Maxwell-Boltzmann distribution function.

What you should put in your pocket

Again, you didn't have to worry about digesting all the details of what we just did. For now, just be aware of the following:

- Entropy $\sigma_{\mathcal{R}}(E)$ is the logarithm of the number of available states as a function of energy E.
- The inverse of temperature τ measures the degree to which entropy changes as the energy in the system is changed but the number of particles N remains fixed:

$$\frac{1}{\tau} \equiv \left(\frac{\partial \sigma}{\partial U}\right)_N$$

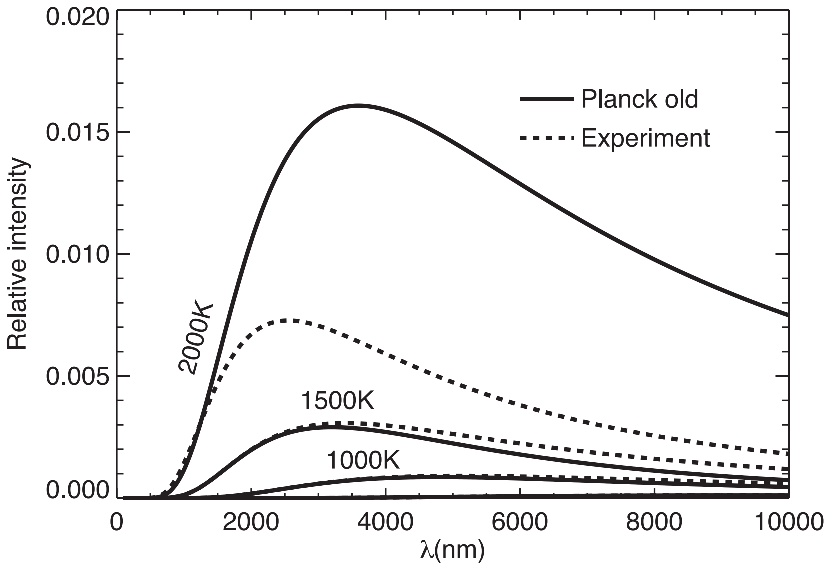
- This is scaled into an absolute temperature T in Kelvin via Boltzmann's constant k_B, so that $\tau = k_B T$.
- The relative likelihood that a system with temperature T will wind up in a particular state with energy E is given by the Maxwell-Boltzmann distribution function which is $\exp[-E/k_B T]$.

Back to Blackbody I

- We now have the two ingredients:
  - → *Density of available states* $\rho(\nu) = \frac{8\pi}{c^3}\nu^2\, d\nu$ from Eq. 1.
  - → *Occupancy of available states* as a function of temperature: $\exp[-E/k_B T]$ from Eq. 8.
- Does this explain blackbody radiation?

$$\rho(\nu, T) = \frac{8\pi}{c^3}\nu^2\ \exp[-c_2\nu/k_B T]\, d\nu. \quad (9)$$

Hey, at 1000 K or 1500 K it's pretty close except in the infrared end!



Blackbody II

Here's another look at a spectrum at 1500 K: (Serway Fig. 3.5):

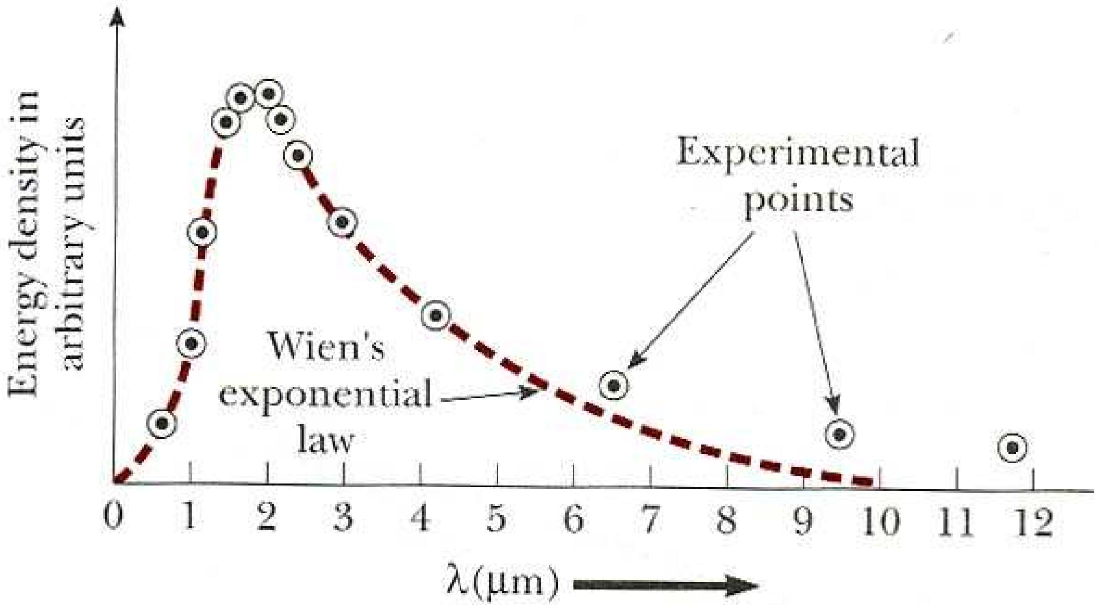


Figure 3.5 Discrepancy between Wien's law and experimental data for a blackbody at 1500 K.

Can't we say that this is good enough? It's only a little bit off at longer infrared wavelengths, where it's not very easy to make a measurement. . .

Blackbody III

Measurements by Heinrich Rubens, Universitæt Berlin, 1900: measure intensity at one infrared wavelength while turning the temperature knob (Serway Fig. 3.6):

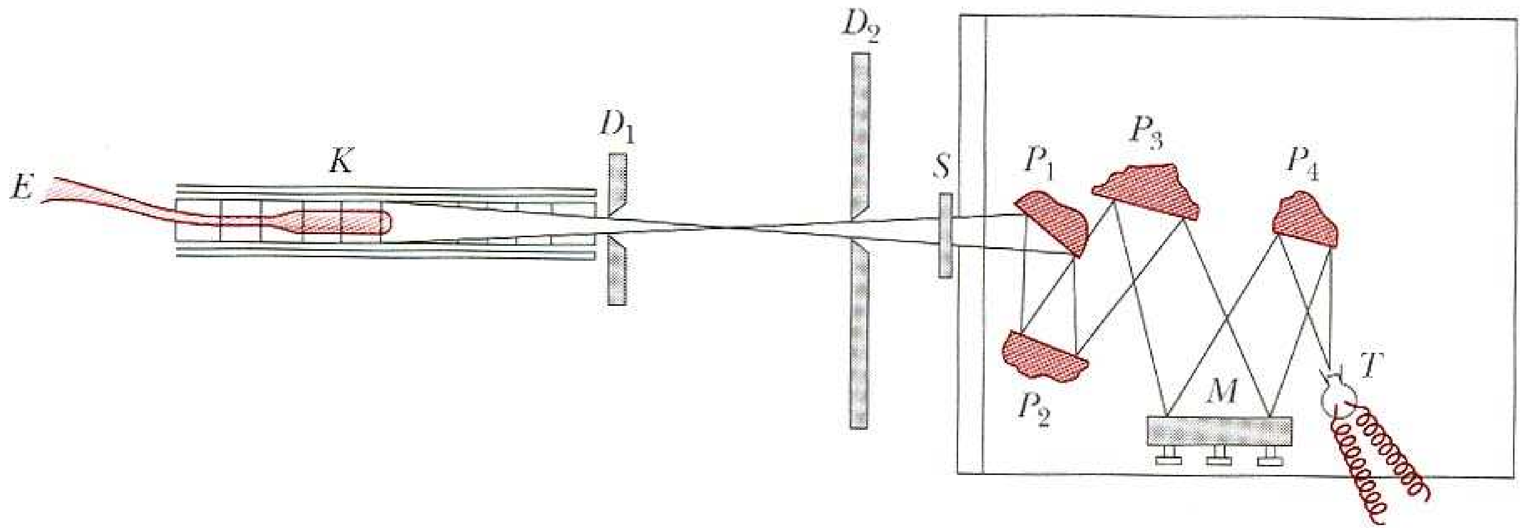


Figure 3.6 Apparatus for measuring blackbody radiation at a single wavelength in the far infrared region. The experimental technique that disproved Wien's law and was so crucial to the discovery of the quantum theory was the method of residual rays (*Restrahlen*). In this technique, one isolates a narrow band of far infrared radiation by causing white light to undergo multiple reflections from alkalide halide crystals (P_1–P_4). Because each alkali halide has a maximum reflection at a characteristic wavelength, quite pure bands of far infrared radiation may be obtained with repeated reflections. These pure bands can then be directed onto a thermopile (T) to measure intensity. E is a thermocouple used to measure the temperature of the blackbody oven, K.

Rubens' measurements at one wavelength versus temperature (Serway Fig. 3-7):

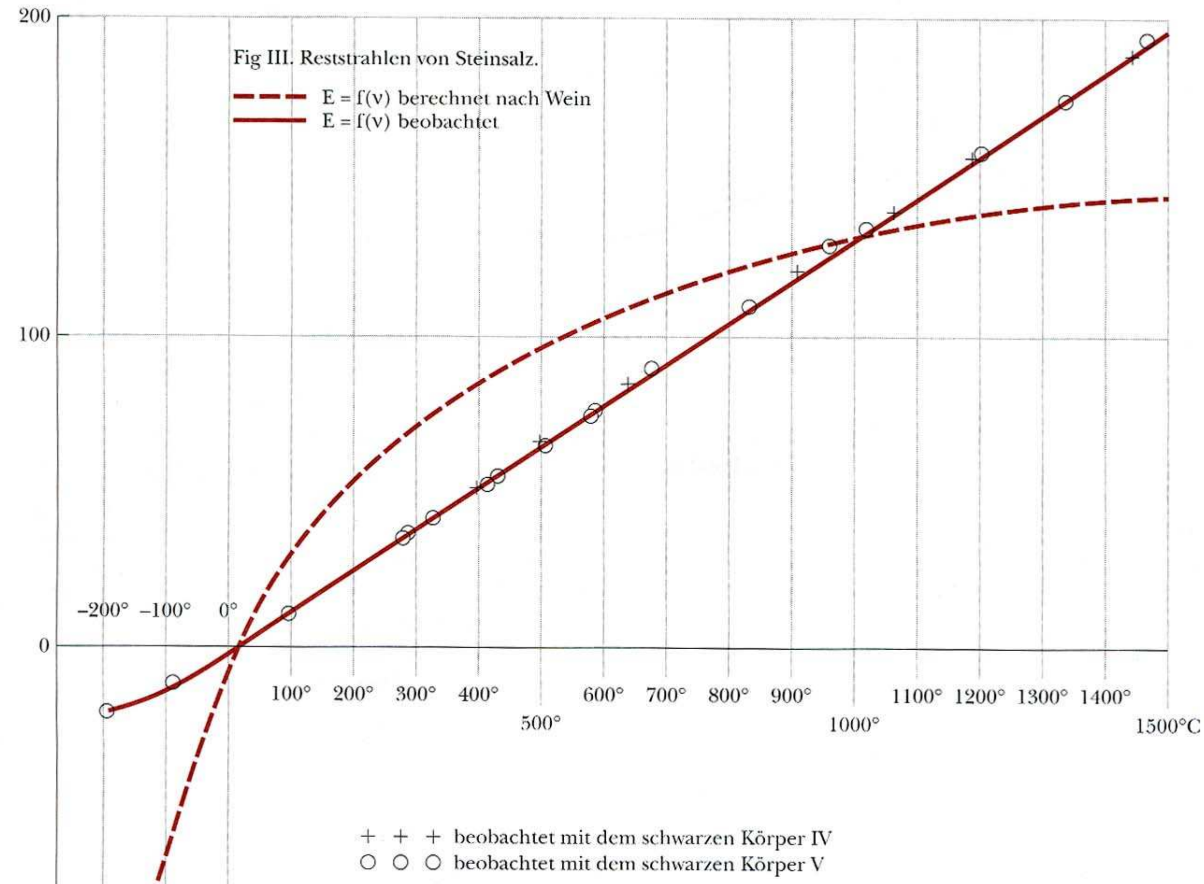


Figure 3.7 Comparison of theoretical and experimental blackbody emission curves at 51.2 μm and over the temperature range of −188° to 1500°C. The title of this modified figure is "Residual Rays from Rocksalt." *Berechnet nach* means "calculated according to," and *beobachtet* means "observed." The vertical axis is emission intensity in arbitrary units. (*From H. Rubens and S. Kurlbaum,* Ann. Physik, *4:649, 1901.*)

Blackbody IV

- Max Planck (1858–1947; Nobel Prize 1918) at Universitæt Berlin had already used statistical mechanics to try to explain the blackbody spectrum.
- His colleague Heinrich Rubens found discrepancies between the Wien law and his group's infrared measurements.
- Oct. 7, 1900: Rubens pays a Sunday afternoon visit to Planck. They discuss the puzzle of Rubens' results. By that evening, Planck has found an answer that explains the experimental data perfectly.

Planck in 1901

Planck's fix

- Assume a quantized energy of

(10)
$$E_n = nh\nu$$

That is, radiation at frequency ν has an energy $h\nu$, and comes quantized in integer multiples n. Fit data to find h; modern value of $h = 6.63 \times 10^{-34}$ Joules·sec is called Planck's constant.

- Average energy per mode is then

$$\begin{aligned}\bar{E} &= \frac{\sum_{\text{modes}} (\text{energy per mode}) \cdot (\text{likelihood of mode})}{\sum_{\text{modes}} (\text{likelihood of mode})} \\ &= \frac{\sum_{n=0}^{\infty} nh\nu \ \exp[-nh\nu/k_B T]}{\sum_{n=0}^{\infty} \exp[-nh\nu/k_B T]} \\ &= \frac{0 + h\nu \exp[-h\nu/k_B T] + 2h\nu \exp[-2h\nu/k_B T] + \ldots}{1 + \exp[-h\nu/k_B T] + \exp[-2h\nu/k_B T] + \ldots} \\ &= h\nu\, y \, \frac{1 + 2y + 3y^2 + \ldots}{1 + y + y^2 + \ldots} \text{ with } y \equiv \exp[-h\nu/k_B T] \qquad (11)\end{aligned}$$

Planck's fix II

- Again, we had

$$\bar{E} = h\nu\, y \,\frac{1 + 2y + 3y^2 + \dots}{1 + y + y^2 + \dots} \text{ with } y \equiv \exp[-h\nu/k_B T]$$

- Now

$$\frac{1 + 2y + 3y^2 + \dots}{1 + y + y^2 + \dots} = \frac{1 + y + y^2 + \dots}{1 + y + y^2 + \dots} + \frac{y + 2y^2 + \dots}{1 + y + y^2 + \dots},$$

so that if we define $A \equiv 1 + 2y + 3y^2 + \dots$ and $B \equiv 1 + y + y^2 + \dots$, Eq. 13 becomes

$$\frac{A}{B} = 1 + y\frac{A}{B} \qquad \text{giving} \qquad (1-y)\frac{A}{B} = 1 \qquad \text{giving} \qquad \frac{A}{B} = \frac{1}{1-y}.$$

The average energy $\bar{E}$ of a mode is then

(12)
$$\bar{E} = h\nu\, y\,\frac{1}{1-y} = \frac{h\nu}{1/y - 1} = \frac{h\nu}{\exp[h\nu/k_B T] - 1}.$$

Planck's fix III

Let's recapitulate:

- We know the *density of available states* from Eq. 1 of $\rho(\nu) = \frac{8\pi}{c^3}\nu^2\, d\nu$.
- We know the combination of *occupancy of available states* multiplied by their energy as $\bar{E} = h\nu/(\exp[h\nu/k_B T] - 1)$.

The product of the two then gives the energy spectrum (*cf.* Serway Eq. 3.9):

$$\rho(\nu) \cdot \bar{E}(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp[h\nu/k_B T] - 1}\, d\nu. \tag{13}$$

This is the celebrated Planck blackbody radiation distribution function for energy per unit volume per unit frequency derived on that Sunday afternoon, Oct. 7, 1900. It fits the experimental data perfectly. Recall our basic assumption was $E_n = nh\nu$: radiation at frequency ν has an energy $h\nu$, and comes quantized in integer multiples n.

Planck's fix IV

How to interpret? Planck himself was not sure. Result was not immediately appreciated; Rayleigh-Jeans theory of Eq. 9

$$\rho(\nu, T) = \frac{8\pi}{c^3}\nu^2 \, \frac{1}{\exp[c_2\nu/k_B T]} \, d\nu.$$

was published *after* Planck published his result of

$$S_\nu = \rho(\nu) \cdot \bar{E}(\nu, T) = \frac{8\pi h}{c^3} \, \nu^3 \, \frac{1}{\exp[h\nu/k_B T] - 1} \, d\nu.$$

To convert to a spectrum in wavelength, we use $\nu = c/\lambda$ which gives $d\nu = -(c/\lambda^2)d\lambda$, so that Eq. 13 becomes

$$S_\lambda = 8\pi hc \, \frac{1}{\lambda^5} \, \frac{1}{\exp[hc/\lambda k_B T] - 1} \, d\lambda. \tag{14}$$

Planck's fix V

- Energy per photon: $E = h\nu = hc/\lambda$. With Planck's constant determined from blackbody spectrum to be $h = 6.63 \times 10^{-34}$ Joules·sec, we can write

$$hc = \frac{(6.626\,069\,3 \times 10^{-34}\ \mathrm{J \cdot s}) \cdot (2.997\,924\,58 \times 10^{8}\ \mathrm{m/sec})}{(1.602\,176\,53 \times 10^{-19}\ \mathrm{J/eV})} \cdot \frac{10^9\ \mathrm{nm}}{\mathrm{m}}$$

$$= 1239.8419\ \mathrm{eV \cdot nm}$$

- Based on physical constants from http://physics.nist.gov/constants which is a web site of NIST (the National Institute of Standards and Technology).
- Consider a 5 mW laser pointer at $\lambda = 680$ nm. Energy of one photon is $E = hc/\lambda = 1239.8419/680 = 1.82$ eV. Photon flux is

$$\frac{(5 \times 10^{-3}\ \mathrm{J/s})}{(1.82\ \mathrm{eV/photon}) \cdot (1.602 \times 10^{-19}\ \mathrm{J/eV})} = 1.71 \times 10^{16}\ \mathrm{photons/s}$$

Wavelength of emission maximum

What's the frequency of the emission maximum?

$$\frac{d}{d\nu} S_\nu = \frac{d}{d\nu}\left[\frac{8\pi h}{c^3}\,\nu^3\,(\exp[h\nu/k_B T]-1)^{-1}\,d\nu\right] = 0$$

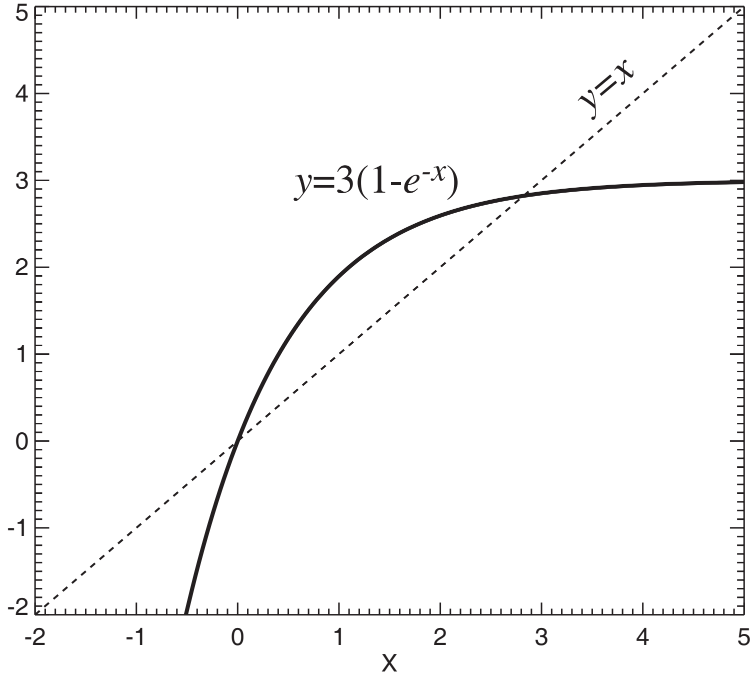
$$\frac{8\pi h}{c^3}\left(3\nu^2(\exp[h\nu/k_BT]-1)^{-1} - \nu^3(\exp[h\nu/k_BT]-1)^{-2}\,\exp[h\nu/k_BT]\,(h/k_BT)\right) = 0$$

$$\left(3 - (\exp[h\nu/k_BT]-1)^{-1}\,\exp[h\nu/k_BT](h\nu/k_BT)\right) = 0$$

$$\begin{aligned}
3 &= \frac{\exp[h\nu/k_BT]}{\exp[h\nu/k_BT]-1}(h\nu/k_BT)\\
3 &= \frac{1}{1-\exp[-h\nu/k_BT]}(h\nu/k_BT)\\
3 &= \frac{x}{1-\exp[-x]} \qquad \text{with} \qquad x \equiv (h\nu/k_BT)\\
3(1-\exp[-x]) &= x
\end{aligned}$$

Wavelength of emission maximum II

We want to find values of x that satisfy $3(1 - e^{-x}) = x$. If we plot x (a straight line with slope=1) and $3(1 - e^{-x})$ we can see the solution points graphically:



This shows that solutions are $x = 0$ and $x \simeq 2.8$. We can pin it down more using Maple:

```
eq:=3*(1-exp(-x))=x;    evalf(solve(eq,x));
    2.821439372, 0.
```

Thus $2.821 = h\nu/k_B T$ or $\lambda_{\text{peak}} = hc/(2.821 k_B T)$.

Total radiated power

The total radiated power is given by

$$\begin{aligned}\int_0^\infty S_\nu &= \int_0^\infty \frac{8\pi h}{c^3}\,\nu^3\,\frac{1}{\exp[h\nu/k_BT]-1}\,d\nu \\ &= \frac{8\pi h}{c^3}\int_0^\infty (h\nu/k_BT)^3(k_BT/h)^3\,\frac{1}{\exp[h\nu/k_BT]-1}\,d(h\nu/k_BT)\,(k_BT/h) \\ &= \frac{8\pi h}{c^3}(k_BT/h)^4\int_0^\infty \frac{x^3}{\exp[x]-1}\,dx \qquad \text{with} \qquad x\equiv h\nu/k_BT\end{aligned}$$

Ask Maple to solve `int(x^3/(exp(x)-1),x=0..infinity);`

This gives $\pi^4/15$, so

$$\int_0^\infty S_\nu = \frac{8\pi^5}{15(hc)^3}\,(k_BT)^4$$

Stefan-Boltzman law from classical thermodynamics!

Classical light

- Recall $c = 1/\sqrt{\mu_0 \epsilon_0}$
- Poynting vector of EM waves gives mean fields of

(15) $$\langle E \rangle = \left(\frac{\mu}{\epsilon}\right)^{1/4} \sqrt{I} \quad \text{and} \quad \langle B \rangle = \left(\mu^3 \epsilon\right)^{1/4} \sqrt{I}$$

for a given irradiance I.

- Focus of a 5 mW, $\lambda = 632$ nm laser to a 10 μm diameter spot: $I = 6.4 \times 10^7$ W/m^2
- Mean electric field: $\langle E \rangle = 1.5 \times 10^5$ V/m. Air sparks at a *static* field of 8×10^5 V/m. Strip electrons from hydrogen: about 10 eV electron binding energy and about 1 Å=10^{-10} m orbital radius, so around 10^{11} V/m required.
- Mean magnetic field: $\langle B \rangle = 5.2 \times 10^{-4}$ Tesla. Earth's magnetic field: 0.3–0.6$\times$10^{-4} Tesla depending on location.

Cathode ray tubes: Philipp Lenard

- Gas discharge lamps (neon lights) were not uncommon. When more highly evacuated, a beam could be seen. Electromagnetic wave or particle beam?
- Lenard builds a tube with a thin aluminum window. Beam can emerge and strike a fluorescent screen. Particle beam!
- By the way, don’t admire Lenard too much: according to Wikipedia, he promoted “German” physics over such “Jewish” physics ideas as those of Einstein, and became Chief of Aryan Physics under the Nazis.

Philipp Lenard (1862–1947; Nobel Prize 1905).

Discovery of the electron: J.J. Thomson

- Careful experiments with Lenard tubes. Must evacuate really well to eliminate electric field "channel" due to gas ionization.
- Found that cathode "rays" can be deflected by both electric and magnetic fields. Negative particles; measured ratio e/m.

J.J. Thomson (1856–1940; Nobel Prize 1906)

More curious properties

- Further experiments by Lenard, *Annalen der Physik* **8**, 169–170 (1902)
- Ultraviolet light can induce a photocurrent, but visible light cannot.
- Classically, one would expect the effect to vary with $\langle E \rangle \propto \sqrt{I}$.
- Experimental fact: current varies linearly with I, and effect requires reaching a threshold short wavelength which depends on cathode material.
- What the. . .?!