Particles in motion

- Towards the end of the previous lecture, we found$E_k = (\gamma - 1)m_0c^2$ (and, for $\beta \ll 1$, $E_k \simeq (1/2)mv^2$) which suggests a new interpretation of the total energy of a particle in motion.
- Assume that a particle at rest has some energy E_0 associated with it:

(1)
$$E = E_0 + E_k.$$

- For $\gamma \gg 1$, we found $E \simeq E_k \simeq \gamma m_0c^2$. Therefore, we make the association $E_0 = m_0c^2$ which allows us to write the total energy as

(2)
$$E = E_0 + E_k = m_0c^2 + (\gamma - 1)m_0c^2$$

or the sum of rest and kinetic energy. We can also write $E = \gamma m_0c^2$ for the total energy.

Rest energy and electron-Volts

- $E_0 = m_0 c^2$ is probably the best-known result of modern physics. "Weigh" particles in natural units for atomic and nuclear physics calculations.
- The electron-Volt, or eV: energy gained by an electron as it experiences an electrostatic potential change of one volt. Work is $W = qV$, giving

$$1 \text{ eV} = 1.6 \times 10^{-19} \, \frac{\text{Coulomb}}{e^- \text{ charge}} \cdot 1 \text{ Volt} = 1.6 \times 10^{-19} \text{ Joule.} \tag{3}$$

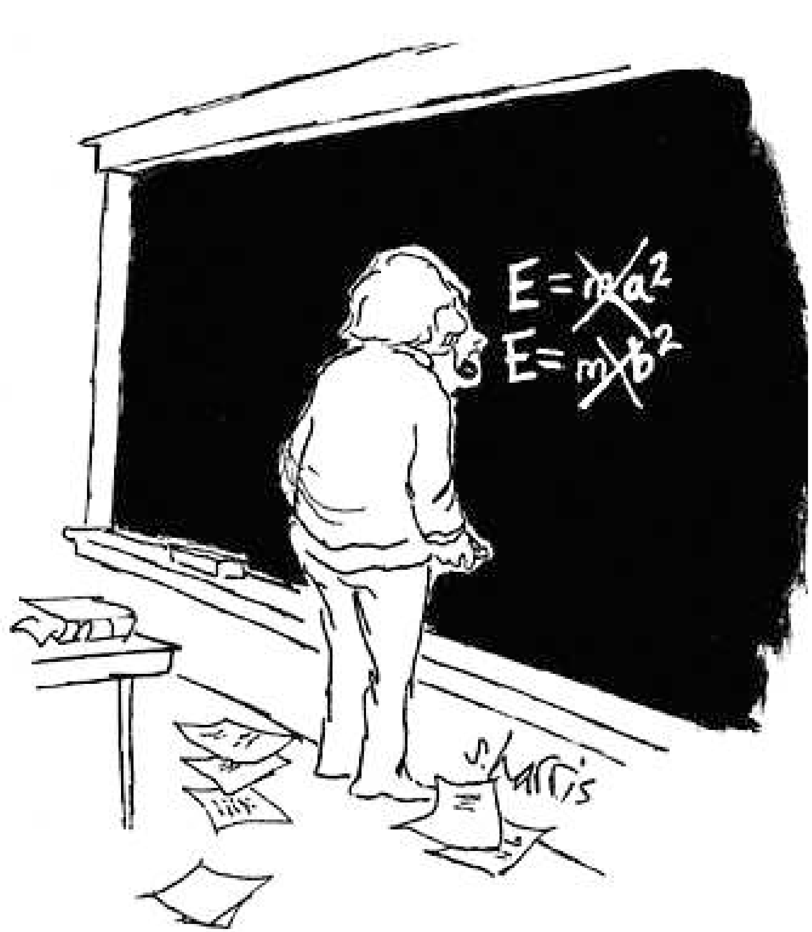
- Proton mass in eV:

$$\begin{aligned} E_0 &= m_0 c^2 = 1.67 \times 10^{-27} \text{ kg} \cdot (3 \times 10^8 \text{ m/sec})^2 \\ &= 1.5 \times 10^{-10} \text{ Joules} \cdot \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ Joules}} \\ &= 939 \times 10^6 \text{ eV.} \end{aligned} \tag{4}$$

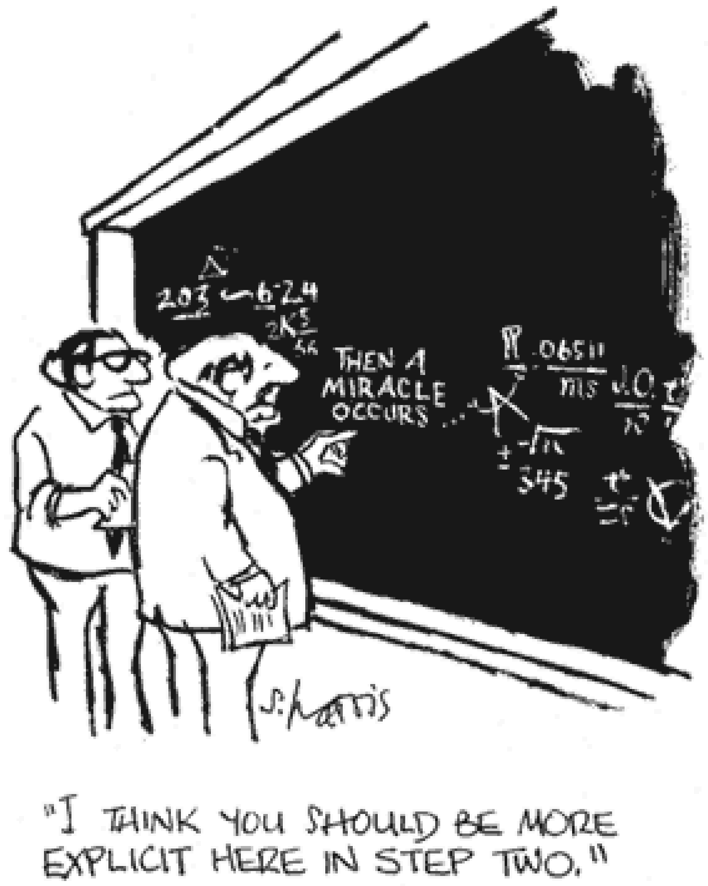
The mass $m_0 = E_0/c^2$ can then be written as $m_0 = 939 \text{ MeV}/c^2$. Sloppy version: "the proton mass is 939 MeV" or "the electron mass is 511 keV."

$E = mc^2$

Two of Sidney Harris' cartoons; from http://www.sciencecartoonsplus.com:



How Einstein arrived at $E = mc^2$



Exam solutions of this sort will not receive credit.

Chemical reactions

- Some chemical bond energies: H-C bond 80.9 kcal/mol, C-N bond 184 kcal/mol, C-O 257 kcal/mol.
- Convert H-C to kJ/mol:

$$80.9\ \frac{\text{kcal}}{\text{mol}} \cdot 4.184\ \frac{\text{kJ}}{\text{kcal}} = 338\ \text{kJ/mol.}$$

- Convert to eV/atom:

$$338\ \frac{\text{kJ}}{\text{mol}} \cdot \frac{10^3\ \text{J}}{\text{kJ}} \cdot \frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}} \cdot \frac{1\ \text{mol}}{6.02 \times 10^{23}} = 3.50\ \text{eV}/\text{bond}$$

- Equating to $m_0 c^2$ gives a fractional mass change for the total of hydrogen ^{1}H plus carbon ^{12}C atoms of

$$\frac{3.50\ \text{eV}}{(1+12) \cdot 940 \times 10^6\ \text{eV}} = 3 \times 10^{-10}$$

- Difficult to detect any mass change due to chemical bonding! **Correspondence principle** in action again.

Relativistic conservation of momentum

- Classically, kinetic energy is $p^2/2m$. Consider p^2 in relativity:

(5)
$$(pc)^2 = (\gamma m_0 vc)^2 = (\gamma\beta m_0 c^2)^2.$$

- If we then use $E_0 = m_0c^2$ and $\beta^2 = 1 - \frac{1}{\gamma^2}$, we obtain

(6)
$$p^2c^2 = \gamma^2(1 - \frac{1}{\gamma^2})E_0^2 = (\gamma^2 - 1)E_0^2 = \gamma^2 E_0^2 - E_0^2.$$

However, since we found before that the total energy is $E = \gamma E_0$, we have $p^2c^2 = E^2 - E_0^2$ or

(7)
$$E^2 = E_0^2 + p^2c^2.$$

- Therefore if $E_k \gg E_0$ we have $E_{k,\text{relativistic}} \simeq pc$.

Mass of the photon

- Mass of something that is allowed to travel at the speed of light? From Eq. 2 of $E = E_0 + E_k = m_0c^2 + (\gamma - 1)m_0c^2$ we get $E = \gamma E_0$ or

(8) $$E_0 = \frac{E}{\gamma} = E\sqrt{1 - v^2/c^2}.$$

  With $v = c$, we get $E_0 = E\sqrt{1-1} = 0$, so that the rest mass m_0 must also be zero.

- Because light must travel with a velocity of c, we therefore conclude that photons have a rest mass of zero.

Let's review

Let's review a few things:

- Relativistic momentum: $p = \gamma m_0 v$.
- $\vec{F} \perp \vec{v}$: $F = \gamma m_0 \vec{a}$.

  This leads to $qBr = \gamma m_0 v$ for velocities v perpendicular to a magnetic field B.
- $\vec{F} \parallel \vec{v}$: $F = \gamma^3 m_0 \vec{a}$
- Total energy is kinetic energy plus energy of rest mass, or

$$E = E_0 + E_k = mc^2 + (\gamma - 1)mc^2 = \gamma mc^2$$

- We can relate total energy with momentum, as we found in Eq. 7:

$$E^2 = E_0^2 + (pc)^2$$

- For photons (which are massless), $E_0 = 0$ so $E = pc$.

Energy–momentum invariant

Rearrange Eq. 7 to give

$$\begin{aligned} \left(\frac{E_0}{c}\right)^2 &= \left(\frac{E}{c}\right)^2 - p^2 \\ &= \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2. \end{aligned} \tag{9}$$

$(E_0/c)^2 = (m_0 c)^2$ has the same value when measured in any inertial frame; therefore so does the quantity on the right hand side of Eq. 9. Thus

$$\left(\frac{E_1}{c}\right)^2 - p_{x,1}^2 - p_{y,1}^2 - p_{z,1}^2 = \left(\frac{E_2}{c}\right)^2 - p_{x,2}^2 - p_{y,2}^2 - p_{z,2}^2 \tag{10}$$

between frames S_1 and S_2. This is really the equivalent of our statement in a previous lecture

$$(ct_1)^2 - x_1^2 - y_1^2 - z_1^2 = (ct_2)^2 - x_2^2 - y_2^2 - z_2^2 \tag{11}$$

which served as the basis for our derivation of the Lorentz transformations!

Momentum transforms

We went from $(ct_1)^2 - x_1^2 - y_1^2 - z_1^2 = (ct_2)^2 - x_2^2 - y_2^2 - z_2^2$ to find transformations for position and time:

$$\begin{aligned} x_2 &= \gamma\,(x_1 - vt_1) \qquad \text{and} \qquad y_2 = y_1 \qquad \text{and} \qquad z_2 = z_1 \\ t_2 &= \gamma\left(t_1 - \frac{\beta}{c}x_1\right) \end{aligned}$$

From Eq. 10 of $(E_1/c)^2 - p_{x,1}^2 - p_{y,1}^2 - p_{z,1}^2 = (E_2/c)^2 - p_{x,2}^2 - p_{y,2}^2 - p_{z,2}^2$ we can find an equivalent Lorentz transform for momentum and energy:

$$\begin{aligned} p_{x,2} &= \gamma\left(p_{x,1} - v\left(\frac{E_1}{c^2}\right)\right) \qquad \text{and} \qquad p_{y,2} = p_{y,1} \qquad \text{and} \qquad p_{z,2} = p_{z,1} \\ E_2 &= \gamma(E_1 - vp_{x,1}). \end{aligned} \tag{12}$$

This is somewhat startling, for it tells us that we need to worry about the Lorentz transformation in considering conservation of energy!

Velocity from momentum and energy

Again by comparing $(ct_1)^2 - x_1^2 - y_1^2 - z_1^2 = (ct_2)^2 - x_2^2 - y_2^2 - z_2^2$

with $(E_1/c)^2 - p_{x,1}^2 - p_{y,1}^2 - p_{z,1}^2 = (E_2/c)^2 - p_{x,2}^2 - p_{y,2}^2 - p_{z,2}^2$

we can also find a particle's velocity v in terms of its total energy E and momentum p using

$$\vec{v} = \frac{\vec{p}}{\gamma m_0} = \frac{\vec{p}}{E/c^2} = \frac{\vec{p}c^2}{E}. \tag{13}$$

Relativisic collision I

A particle A with rest mass m_0 and velocity $v_A = 0.80c$ in the $\hat{x}$ direction in frame S_1 collides with an initially-stationary (in S_1) particle B with rest mass $2m_0$. Note that for $\beta = 4/5$ we can find $\gamma = \frac{5}{3}$. In the frame S_1, we then use $p_{1,y} = \gamma m_0 v_{2,y}$, and $E = \gamma m_0 c^2$ to find

$$\begin{aligned}
p_{x,A,1} &= \gamma m_0 v_{x,A} = \frac{5}{3} m_0 \frac{4}{5} c = \frac{4}{3} m_0 c. \\
p_{y,A,1} &= p_{z,A,1} = 0 \\
E_{A,1} &= \gamma m_0 c^2 = \frac{5}{3} m_0 c^2
\end{aligned}$$

for particle A, and

$$\begin{aligned}
p_{x,B,1} &= \gamma m_0 v_{x,B} = 0 \\
p_{y,B,1} &= p_{z,B,1} = 0 \\
E_{B,1} &= \gamma m_0 c^2 = 1(2m_0)c^2
\end{aligned}$$

for particle B.

Relativistic collision II

The total energy in frame S_1 is then

$$E_1 = E_{A,1} + E_{B,1} = \left(\frac{5}{3} + 2\right) m_0 c^2 = \frac{11}{3} m_0 c^2,$$

or $E_1 = 3.67 m_0 c^2$.

To find the velocity v needed to put us in the center-of-momentum frame S_2, we want to set the net momentum $p_{x,2} = \gamma \left(p_{x,1} - v(E/c^2)\right)$ to zero:

$$\begin{aligned} p_{x,A,2} + p_{x,B,2} = 0 &= \gamma\left[\left(p_{x,A,1} - v(E_{A,1}/c^2)\right) + \left(p_{x,B,1} - v(E_{A,1}/c^2)\right)\right] \\ &= \frac{\left(\frac{4}{3} m_0 c - \frac{5}{3} m_0 v\right) + \left(0 - 2 m_0 v\right)}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

The result of the final line will be zero if the numerator on the final line is zero, or $\left(\frac{4}{3}c - \frac{5}{3}v - 2v\right) m_0 = 0$ from which we obtain a velocity v of S_2 relative to S_1 of $v = (4/11)c$.

Relativistic collision III

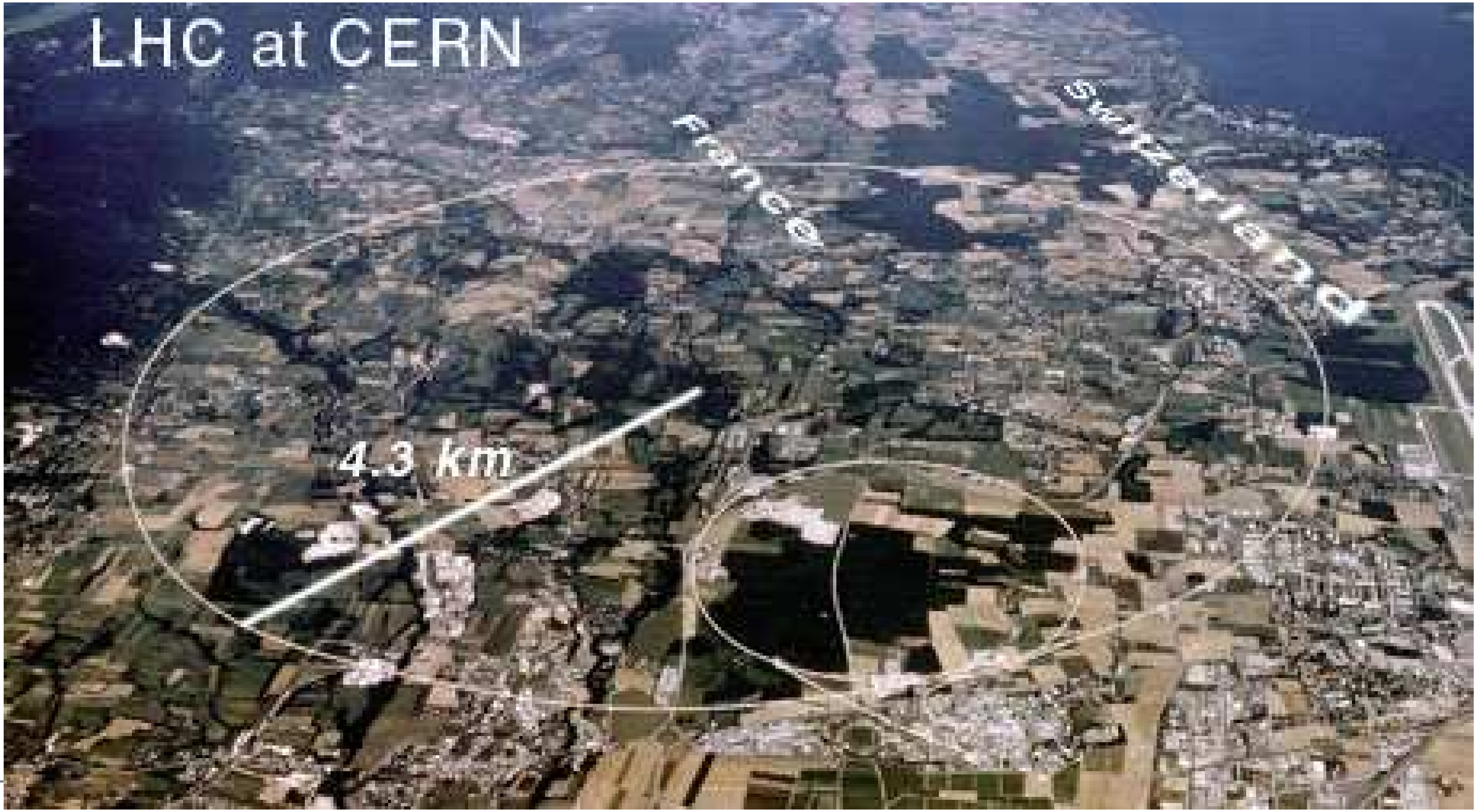
The total energies of the individual particles in S_2 can be found using the result of $E_2 = \gamma(E_1 - vp_{x,1})$ to be

$$
\begin{aligned}
E_{A,2} &= \frac{E_{A,1} - vp_{x,A,1}}{\sqrt{1 - v^2/c^2}} \\
&= \frac{\frac{5}{3}m_0c^2 - (\frac{4}{11}c)(\frac{4}{3}m_0c)}{\sqrt{1 - \left(\frac{4}{11}\right)^2}} = 1.27m_0c^2 \\
E_{B,2} &= \frac{E_{B,1} - vp_{x,B,1}}{\sqrt{1 - v^2/c^2}} \\
&= \frac{2m_0c^2 - 0}{\sqrt{1 - \left(\frac{4}{11}\right)^2}} = 2.15m_0c^2,
\end{aligned}
$$

or $E_2 = 3.24m_0c^2$ (compare with $E_1 = 3.67m_0c^2$), so the total energy needed to create the collision in the center-of-mass frame is less than in the fixed-target frame. If you want to get a certain total energy in a collision, you're better off producing it in the center-of-mass frame when relativistic effects are considered (hence the use of collider accelerators in high energy physics experiments).

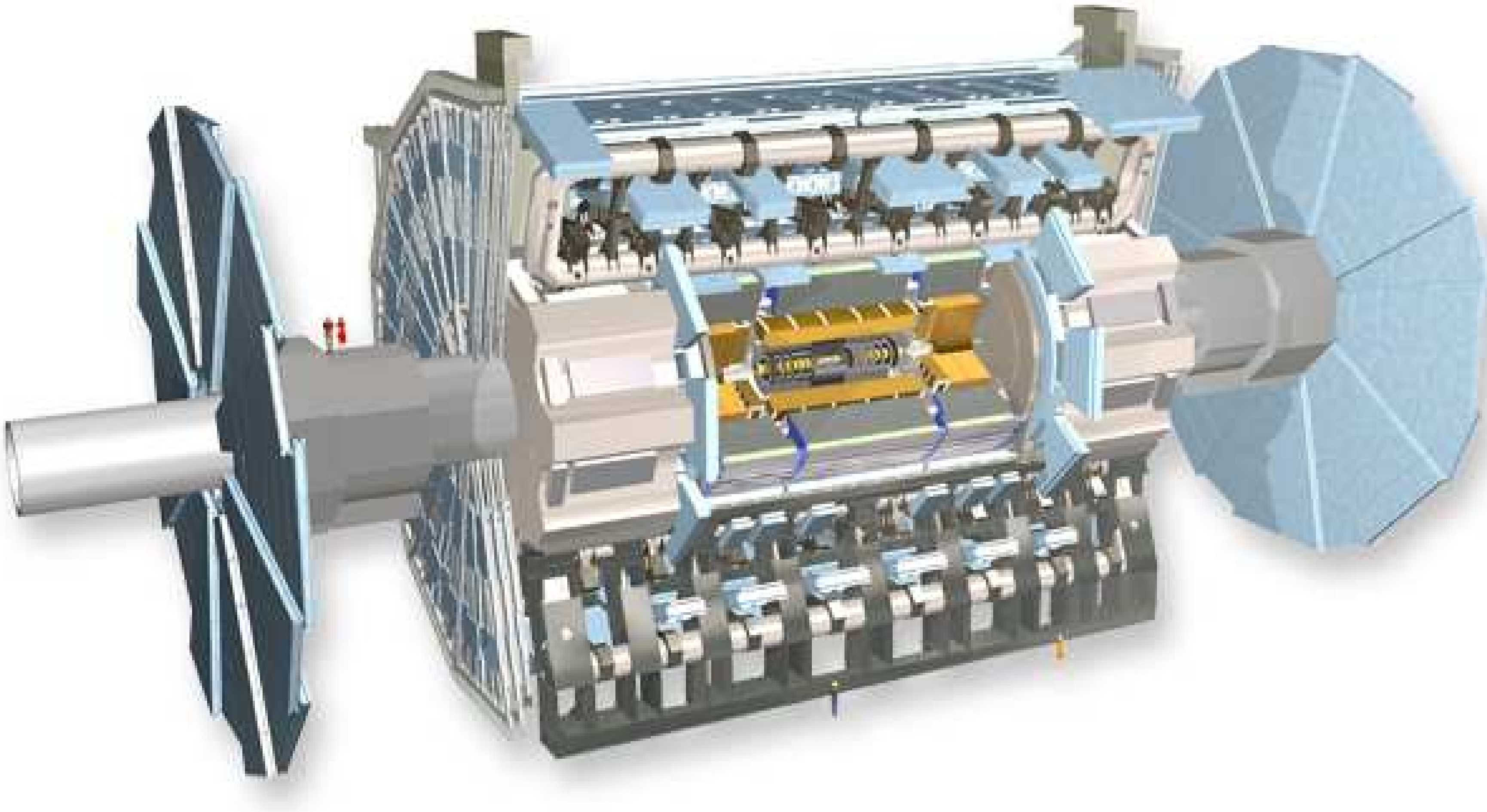
Large Hadronic Collider or LHC

The next (last?) large accelerator for subatomic particle physics is nearing completion (Nov. 2007?) at CERN (http://www.cern.ch; originally stood for Conseil Européen pour la Recherche Nucléaire) in Geneva. 7 TeV (7×10^{12} eV) protons against 7 TeV anti-protons.



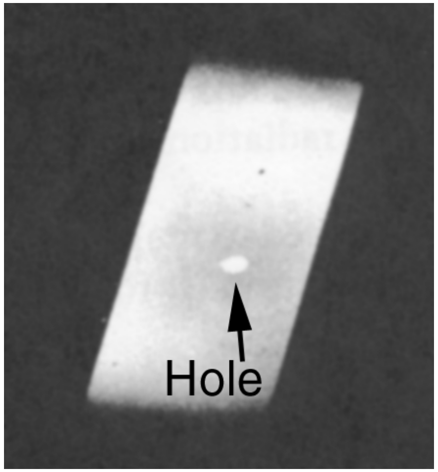
ATLAS at the LHC

Several Stony Brook faculty are involved in experiments that will use this detector for capturing proton—anti-proton collisions at LHC. Notice the size of the people in this computer rendering?



Blackbody radiation

- Prosaic beginning to our story: how hot is a white-hot oven? Thomas Wedgwood (Wedgwood China still exists! http://www.wedgwood.com) notices in 1792 that all objects in the firing ovens shown the same shade of red no matter what their color is when cool.
- Classical thermodynamics has already uncovered Stefan's law which says that the total radiated power is $\propto T^4$. Note that heat transfer at room temperature is dominated by conduction and convection, not radiation. . .
- Heat a cavity to uniform temperature; observe radiation spectrum through a hole so small that we can neglect its cooling effects.
- How can we predict the spectrum?

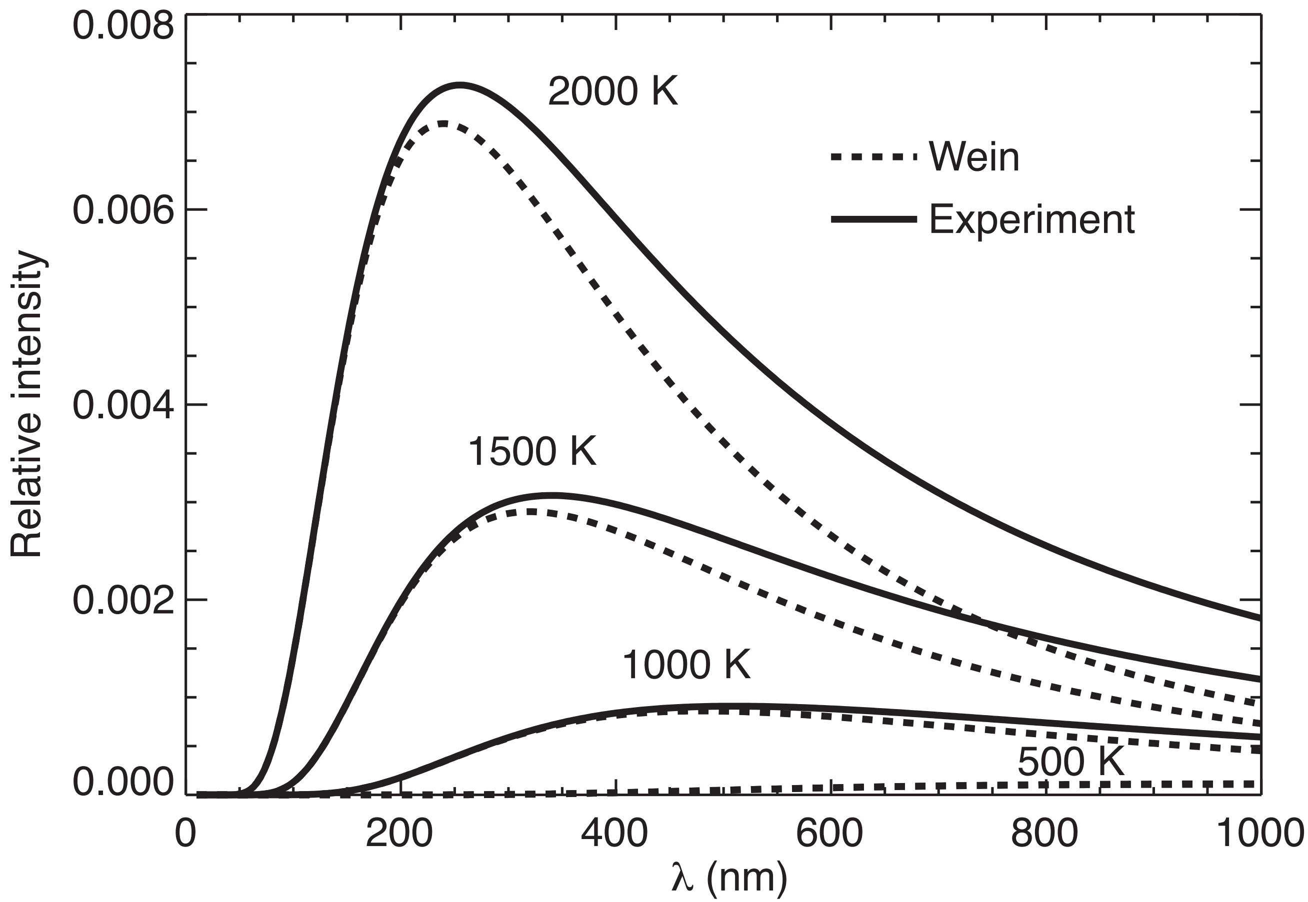


Measuring a black body

What does the spectrum look like?

Empirical fit by Wien in 1896:

(14) $$\rho(\nu, T) = c_1 \nu^3 \exp[-c_2 \nu / T]$$



Elements of the calculation

- Our calculation will push us into a first look at statistical mechanics. We will want to know of two things:
  - → *Density of available states* $g(E)$, which in this case will deal with the number of possible configurations of photons.
  - → *Probability of occupying available states* $f(E)$, which in this case will deal with how many photons are likely to be in each of various states.
- You already know that electrons in atoms reside in discrete orbitals with particular energies. When you heat an atom up (add energy), which excited states are the electrons likely to be in?
- A crude example:
  - → *Density of states* $g(E)$ says we have A plates of raw liver, and B bowls of ice cream available.
  - → *Occupancy of states* $f(E)$ tells us that if we put N hungry kids in the room, what the occupancy $f(A)$ of state A is (how many kids will be eating raw liver), and what the occupancy $f(B)$ is (how many kids will be eating ice cream). You can guess that if $N \leq B$ we will find $f(B) = N$ and $f(A) = 0 \ldots$

States in blackbody radiation

- Assume that blackbody is a cavity of dimension $L_x \times L_y \times L_z$.
- If cavity is a conductor, must have nodes of electromagnetic waves at cavity walls.
- Therefore there must be an integer number of half-wavelengths along each dimension, or $n_x(\lambda/2) = L_x$.
- Since $c = \lambda\nu$ we can rearrange to obtain

  $$\nu_x = n_x \frac{c}{2L_x} \tag{15}$$

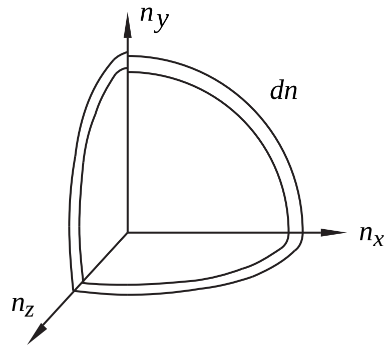
  for allowed frequencies in the $\hat{x}$ direction. Since L_x can be continuously varied, so can ν_x, but n_x must be an integer.
- These are standing waves rather than traveling waves, so n_x is always positive.

Density of available states

- In three dimensions:

(16) $$n = \sqrt{n_x^2 + n_y^2 + n_z^2}.$$

- Density of allowed states for a range of modes n to $n + dn$: a shell in the positive octant.



- Volume of a sphere is $(4/3)\pi r^3$, so volume between n and $n + dn$ is

$$\begin{aligned}
\frac{4}{3}\pi\left[(n+dn)^3 - n^3\right] &= \frac{4}{3}\pi n^3\left[\left(1+\frac{dn}{n}\right)^3 - 1\right] \\
&\simeq \frac{4}{3}\pi n^3\left[1 + 3\frac{dn}{n} - 1\right] \\
&= 4\pi n^2\, dn.
\end{aligned}$$

Density of states II

- We found that the shell of a sphere had $4\pi n^2\, dn$ available states. However, since we can only have positive n_x, n_y, and n_z, only one octant of sphere corresponds to physical states, so multiply by 1/8.
- Light can exist in two orthogonal polarizations, so multiply by 2.
- From Eq. 15 of $\nu_x = n_x(c/2L_x)$ we obtain

(17) $$d(n_x) = d\left(\frac{2L_x}{c}\nu_x\right)$$

- Absolute density of available states is thus

(18) $$\left(\frac{1}{8}\right)\cdot(2)\cdot(4\pi n^2)\, dn = \pi\left(\frac{2L}{c}\right)^3 \nu^2\, d\nu = V\,\frac{8\pi}{c^3}\nu^2\, d\nu.$$

- Volume-normalized result $\rho(\nu)$ (Planck, 1897) is then

(19) $$\rho(\nu) = \frac{8\pi}{c^3}\nu^2\, d\nu.$$