Review again: special relativity

Einstein's postulates:

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light in free space has the same value $c = 1/\sqrt{\mu_0 \epsilon_0}$ in all inertial reference frames.

Net transformation between coordinate systems is

$$x_2 = \gamma(x_1 - vt_1) \qquad [\text{also } x_1 = \gamma(x_2 + vt_2)] \tag{1}$$

$$y_2 = y_1 \tag{2}$$

$$z_2 = z_1 \tag{3}$$

$$t_2 = \gamma\left(t_1 - \frac{\beta}{c}x_1\right) \qquad [\text{also } t_1 = \gamma\left(t_2 + \frac{\beta}{c}x_2\right)] \tag{4}$$

with $\beta \equiv v/c$ and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{5}$$

More relativity

- We learned about time dilation: $t' = \gamma t_0$.
- We learned about distance contraction: $\ell' = l_0/\gamma$.
- We learned about velocity transformations:

$$v_{2,x} = \frac{v_{1,x} - v}{1 - \frac{v\,v_{1,x}}{c^2}} \tag{6}$$

$$v_{2,y} = \frac{v_{1,y}}{\gamma\left[1 - \frac{v\,v_{1,x}}{c^2}\right]} \tag{7}$$

$$v_{2,z} = \frac{v_{1,z}}{\gamma\left[1 - \frac{v\,v_{1,x}}{c^2}\right]} \tag{8}$$

- We learned about the relativistic Doppler shift, with $\theta = 0$ corresponding to moving directly away from us:

$$\nu' = \frac{\nu_0}{\gamma(1 + \beta\cos\theta)}$$

- We learned about the Hubble constant, and estimates of the age of the universe.

Relativistic momentum

- Conservation of momentum involves constant center of mass with no external forces, and bookkeeping on how this changes between different inertial frames.
- Special relativity tells us to be more careful in shifting between different inertial frames.
- Let's pick our frame S_1 to have $v_{1,x} = 0$ and $p_{1,x} = 0$. The frame S_2 moves at velocity v in the $\hat{x}$ direction relative to frame S_1.
- Orthogonal to $\hat{x}$: use velocity transformation of Eq. 6 of $v_{2,y} = v_{1,y}/\gamma[1 - v\, v_{1,x}/c^2]$ to give

(9) $$p_{2,y} = m_0 v_{2,y} = m_0 \frac{v_{1,y}}{\gamma},$$

giving $p_{1,y} = m v_{1,y} = \gamma p_{2,y}$ or

(10) $$p_{1,y} = \gamma m_0 v_{2,y}.$$

Inertial mass m_0 of particle in frame S_2 looks to us in S_1 as if it has increased by factor γ.

- If we brought particle to rest in our frame we would measure the same "rest mass" m_0.

Relativistic forces: $\vec{F} \perp \vec{v}$

- Force is defined as the change in momentum per time:

$$\vec{F} \equiv \frac{d(m_0\gamma\vec{v})}{dt} = m_0\gamma\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt} + \gamma\vec{v}\frac{dm_0}{dt} \tag{11}$$

  Rest mass is the rest mass, so $dm_0/dt = 0$.

- Consider case when force is always perpendicular to velocity (for example, charged particle in a magnetic field). Now $\vec{F} \perp \vec{v}$. No velocity and therefore no motion along $\vec{F}$ direction so no work!

- Direction of $\vec{v}$ changes but magnitude does not, so $d\gamma/dt = 0$. We are therefore left with

$$\vec{F}_\perp = m_0\gamma\frac{d\vec{v}}{dt} = m_0\gamma\vec{a} \qquad (\text{for } \vec{F} \perp \vec{v}) \tag{12}$$

Particle in a magnetic field

- Consider a charged particle moving perpendicular to a magnetic field, or $\vec{v} \perp \vec{B}$.
- The Lorentz force is $F = q\vec{v} \times \vec{B}$, so the force is perpendicular to the direction of motion.
- We end up with circular motion! To have circular motion, something must provide a centripetal acceleration of $a = v^2/r$:

$$\vec{F}_{\perp} = \gamma m_0 \vec{a} = -\gamma m_0 \frac{v^2}{r} \tag{13}$$

so we have have

$$qvB = m_0 \gamma \frac{v^2}{r} \quad \text{or} \quad r = \frac{\gamma m_0 v}{qB} \tag{14}$$

which was verified experimentally back in 1909.

Finding $\gamma\beta$

Again, we have $r = \gamma m_0 v/(qB)$ or $\gamma\beta = qrB/(m_0 c)$. What if we know q, r, m_0, and B, and want to find β? Let

$$
\begin{aligned}
x &\equiv \gamma\beta = \frac{\beta}{\sqrt{1-\beta^2}} \\
x^2 &= \frac{\beta^2}{1-\beta^2} \qquad \Rightarrow \qquad (1-\beta^2)x^2 = \beta^2 \\
x^2 &= \beta^2(1+x^2)
\end{aligned}
$$

(15) giving $$\beta = \frac{x}{\sqrt{1+x^2}} = \frac{\gamma\beta}{\sqrt{1+(\gamma\beta)^2}}$$

so we can find β if we know $\gamma\beta$. In this case we know $\gamma\beta = qrB/(m_0 c)\ldots$

$\vec{F} \parallel \vec{v}$

- Now particle speed and thus γ will *not* be constant. Return to Eq. 11 with $dm_0/dt = 0$:

(16) $$\vec{F} \equiv \frac{d(m_0\gamma\vec{v})}{dt} = m_0\gamma\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt}$$

Again, $d\vec{v}/dt = \vec{a}$. Calculate $d\gamma/dt$:

(17) $$\frac{d}{dt}(1 - v^2/c^2)^{-1/2} = -\frac{1}{2}(1 - v^2/c^2)^{-3/2}\,(-2)\frac{\vec{v}}{c^2}\,\frac{d\vec{v}}{dt} = \gamma^3\,\frac{\vec{v}}{c^2}\vec{a}$$

because it's only when we have the square of velocity that we lose information on its direction.

- Returning to Eq. 16 we now have

(18) $$\vec{F}_{\|} = m_0\gamma\vec{a} + m_0\vec{v}\,\gamma^3\,\frac{\vec{v}}{c^2}\vec{a} = m_0\gamma\vec{a}\left(1 + \gamma^2\,\frac{v^2}{c^2}\right)$$

$$= m_0\gamma\vec{a}\left(1 + \frac{v^2}{c^2 - v^2}\right) = m_0\gamma\vec{a}\left(\frac{c^2 - v^2}{c^2 - v^2} + \frac{v^2}{c^2 - v^2}\right)$$

(19) $$= m_0\gamma\vec{a}\,\frac{c^2}{c^2 - v^2} = m_0\gamma\vec{a}\,\frac{1}{1 - v^2/c^2} = m_0\gamma^3\vec{a}.$$

Kinetic energy

- To repeat, we have $F_\perp = \gamma m_0 a$ (Eq. 12), and $F_{||} = \gamma^3 m_0 a$ (Eq. 19).
- We now know force along direction of frame shift $\vec{F} \parallel \vec{v}$. Calculate kinetic energy from work done to move particle:

$$E_k = \int_0^{x'} F_{||}\, dx = \int_0^{x'} m_0 \gamma^3 a\, dx. \tag{20}$$

- Now

$$a\, dx = \frac{dv}{dt}\, dx = \frac{dx}{dt}\, dv = v\, dv, \tag{21}$$

so we can write the kinetic energy as

$$E_k = \int_0^{v'} \gamma^3 m_0 v\, dv = m_0 \int_0^{v'} \frac{v}{(1 - v^2/c^2)^{3/2}}\, dv. \tag{22}$$

Kinetic energy II

- Again we had from Eq. 22

$$E_k = m_0 \int_0^{v'} \frac{v}{(1 - v^2/c^2)^{3/2}}\, dv.$$

Define $A \equiv c^2(1 - v^2/c^2)^{-1/2}$ so that

(23) $$dA = c^2(-1/2)(1 - v^2/c^2)^{-3/2}(-2v/c^2)\, dv = \frac{v}{(1 - v^2/c^2)^{3/2}}\, dv.$$

- Therefore, we can recognize Eq. 22 as $\int dA = A$ and obtain

(24) $$\begin{aligned} E_k &= \frac{m_0 c^2}{(1 - v^2/c^2)^{1/2}} \Big|_0^{v'} \\ &= m_0 c^2 \left(\frac{1}{(1 - v'^2/c^2)^{1/2}} - 1 \right) \\ &= (\gamma - 1) m_0 c^2. \end{aligned}$$

Correspondence principle and kinetic energy

- In classical limit $v \ll c$, we found found $\gamma \simeq 1 + \frac{1}{2}\frac{v^2}{c^2}$.
- Therefore the classical limit of relativistic kinetic energy is

$$E_k \simeq \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) m_0 c^2 = \frac{1}{2} m_0 v^2 \tag{25}$$

  as expected (**correspondence principle**).
- In the highly relativistic limit of $\gamma \gg 1$, we instead obtain

$$E_k \simeq \gamma m_0 c^2. \tag{26}$$

Particles in motion

- $E_k = (\gamma - 1)m_0c^2$ suggests a new interpretation of the total energy of a particle in motion.
- Assume that a particle at rest has some energy E_0 associated with it:

$$E = E_0 + E_k. \tag{27}$$

- For $\gamma \gg 1$, we found $E \simeq E_k \simeq \gamma m_0 c^2$. Therefore, we make the association $E_0 = m_0c^2$ which allows us to write the total energy as

$$E = E_0 + E_k = m_0c^2 + (\gamma - 1)m_0c^2 \tag{28}$$

or the sum of rest and kinetic energy. We can also write $E = \gamma m_0 c^2$ for the total energy.

Rest energy and electron-Volts

- $E_0 = m_0 c^2$ is probably the best-known result of modern physics. “Weigh” particles in natural units for atomic and nuclear physics calculations.
- The electron-Volt, or eV: energy gained by an electron as it experiences an electrostatic potential change of one volt. Work is $W = qV$, giving

$$1 \text{ eV} = 1.6 \times 10^{-19} \frac{\text{Coulomb}}{e^- \text{ charge}} \cdot 1 \text{ Volt} = 1.6 \times 10^{-19} \text{ Joule.} \tag{29}$$

- Proton mass in eV:

$$\begin{aligned} E_0 &= m_0 c^2 = 1.67 \times 10^{-27} \text{ kg} \cdot (3 \times 10^8 \text{ m/sec})^2 \\ &= 1.5 \times 10^{-10} \text{ Joules} \cdot \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ Joules}} \\ &= 939 \times 10^6 \text{ eV.} \end{aligned} \tag{30}$$

The mass $m_0 = E_0/c^2$ can then be written as $m_0 = 939 \text{ MeV}/c^2$. Sloppy version: “the proton mass is 939 MeV” or “the electron mass is 511 keV.”

Chemical reactions

- Some chemical bond energies: H-C bond 80.9 kcal/mol, C-N bond 184 kcal/mol, C-O 257 kcal/mol.
- Convert H-C to kJ/mol:

$$80.9\ \frac{\text{kcal}}{\text{mol}} \cdot 4.184\ \frac{\text{kJ}}{\text{kcal}} = 338\ \text{kJ/mol}.$$

- Convert to eV/atom:

$$338\ \frac{\text{kJ}}{\text{mol}} \cdot \frac{10^3\ \text{J}}{\text{kJ}} \cdot \frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}} \cdot \frac{1\ \text{mol}}{6.02 \times 10^{23}} = 3.50\ \text{eV}/\text{bond}$$

- Equating to $m_0 c^2$ gives a fractional mass change for electron of

$$\frac{3.50\ \text{eV}}{511 \times 10^3\ \text{eV}} = 7 \times 10^{-6}$$

- Difficult to detect any mass change due to chemical bonding! **Correspondence principle** in action again.