Review: Special relativity

Einstein's postulates:

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light in free space has the same value $c = 1/\sqrt{\mu_0 \epsilon_0}$ in all inertial reference frames.

We then sought a fudge factor γ to fix Galilean relativity so that it is consistent with the above. We had frame S_2 moving at a speed v in the $\hat{x}$ direction relative to frame S_1, and we sought to find γ such that

$$x_2 = \gamma(x_1 - vt_1).$$

The Lorentz transformation

We found a net transformation between coordinate systems of

$$
\begin{aligned}
x_2 &= \gamma\,(x_1 - vt_1) && (1)\\
y_2 &= y_1 && (2)\\
z_2 &= z_1 && (3)\\
t_2 &= \gamma\left(t_1 - \frac{\beta}{c}x_1\right) && (4)\\
\text{with}\quad \gamma &\equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1\sqrt{1-\beta^2} && (5)
\end{aligned}
$$

This transformation was first worked out by H. Lorentz in 1904, but in the assumption of a moving æther; as a result, Einstein gets the credit for getting the physics right. Important mathematical contributions later added by Poincaré and Minkowski.

Inverse Lorentz transformations

Since observer at rest in S' or S_2 sees S or S_1 moving at $-v$, the inverse transformations are

$$x_1 = \gamma\,(x_2 + vt_2) \tag{6}$$

$$y_1 = y_2 \tag{7}$$

$$z_1 = z_2 \tag{8}$$

$$t_1 = \gamma\left(t_2 + \frac{\beta}{c}x_2\right) \tag{9}$$

The Lorentz factor γ

- We showed before that

  $$\gamma = (1-\beta^2)^{-1/2} \simeq 1 + \frac{1}{2}\beta^2 \text{ for } \beta \ll 1. \tag{10}$$

  In the limit $\beta \longrightarrow 0$, we have $\gamma \longrightarrow 1$ and Eqs. 1 and 4 both revert to the Galilean results. This illustrates the all-important **correspondence principle** with classical physics.

- If we instead consider the high frequency limit $\beta \longrightarrow 1$, we can write $\beta = 1 - \epsilon$:

  $$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-(1-\epsilon)^2}} = \frac{1}{\sqrt{1-1+2\epsilon-\epsilon^2}} \\ &= \frac{1}{\sqrt{2\epsilon-\epsilon^2}} \simeq \frac{1}{\sqrt{2\epsilon}} \end{aligned} \tag{11}$$

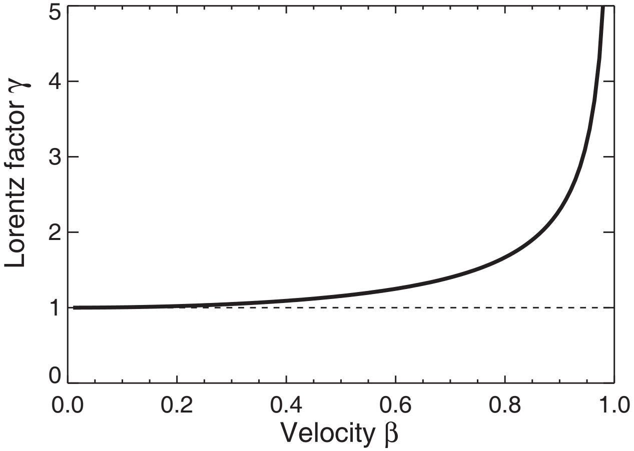
  Why bother? Well, try calculating γ with $\beta = 10^{-7}c$, or with $\epsilon = 10^{-7}$, and see how accurate your result is.

More on the Lorentz factor γ

Given γ, we can solve for β as

$$\beta^2 = 1 - \frac{1}{\gamma^2}. \tag{12}$$

And here’s a plot once again of γ versus β:



Relativistic time dilation

- Fastest way to sense someone’s clock ticks: speed of light/radio.
- We are at rest in frame 1; clock is moving at v in frame 2.
- Light pulses emitted in clock’s frame at t_2 and then t_2', so $T_2 = t_2' - t_2$.
- Light pulses are emitted from same point on clock so $x_2 = x_2'$.
- To us (using Eq. 4):

$$t_1 = \gamma(t_2 + \frac{\beta}{c}x_2) \qquad \text{and} \qquad t_1' = \gamma(t_2' + \frac{\beta}{c}x_2')$$

- Our time interval T_1:

$$T_1 = t_1' - t_1 = \gamma\left((t_2' - t_2) + \frac{\beta}{c}(x_2' - x_2)\right) = \gamma T_2 \tag{13}$$

We see time dilated by γ relative to the moving frame. This is often written as $t' = \gamma t_0$.

The twin paradox

- At age 20, Sluggo stays home, while his twin Speedo travels to the Planet of the Apes, 10 light years from Earth, at $v = 0.5c$. Speedo then returns. Ignore crushing accelerations. . .
- To Sluggo, Speedo's journey took $t = x/v = 20$ years there, and 20 years back, or 40 years total.
- Sluggo's clock is dilated by γ relative to Speedo's; since $\gamma = 1/\sqrt{1 - (0.5c/c)^2} = 1.15$, Speedo's clock ran for $40/1.15 = 34.6$ years.
- Upon Speedo's return, Sluggo is $20 + 40 = 60$ years old, while Speedo is $20 + 34.6 = 54.6$ years old!

The twin paradox: experimental verification

Abstract, right? It's been tested! Atomic clocks have been flown around the world on jets. Time discrepancies in nanoseconds:

Eastward: measured -59 ± 10 nsec, predicted -40 ± 23 nsec

Westward: measured 273 ± 7 nsec, predicted 275 ± 21 nsec

See Hafele and Keating, *Science* **177**, 166 and 168 (1972). Note that the calculation involves both special relativity (plane going at a velocity relative to observer on earth's surface) and general relativity (acceleration from plane ascending and descending, and also from weaker gravitational attraction at altitude).

Time travel?

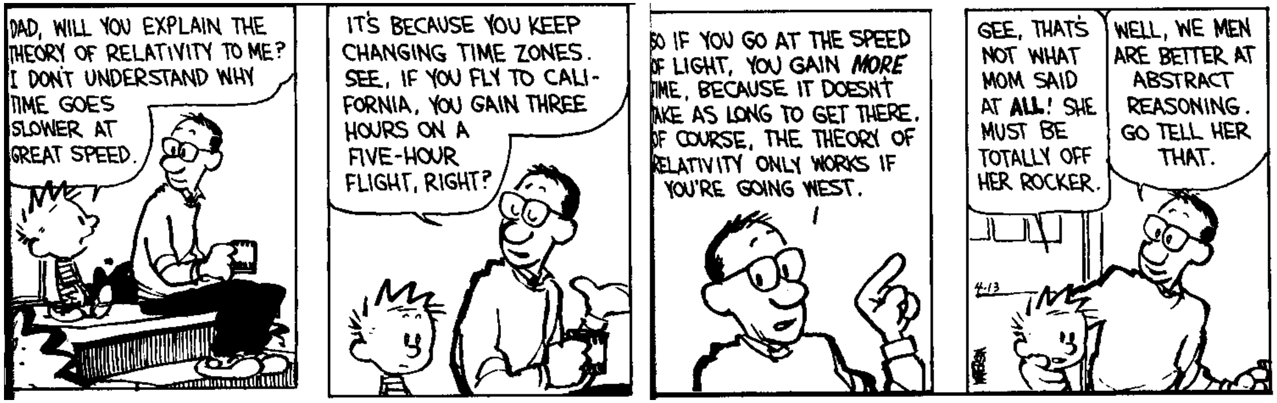
> There was a young lady named Bright,
> Whose speed was far faster than light;
>   She set out one day
>   In a relative way,
> And returned on the previous night

Attributed to Arthur Buller in P. Davies, “Wormholes and time machines,” *Sky & Telescope* **83**, 20 (January 1992).

> Of course, as a physics teacher I tell my students that faster-than-light travel is impossible, but that’s just to crush their spirits.

Submitted by LaNelle Ohlhausen to
`http://www.improb.com/airchives/miniair/twentieth-century/MINI9703`
which is the 1997-03 issue of the *Annals of Improbable Research*

Calvin and Hobbes



Thanks to Sam Watterson. . .

Length contraction

- We are at rest in frame 1; measuring rod with ends separated by $L_2 = x_2' - x_2$ is moving at v in frame 2.
- We take a photograph of the rod at one instant of time, so for us $t_1 = t_1'$.
- We measure a distance $L_1 = x_1' - x_1$.
- Using Eq. 1, we can now relate the two lengths:

$$x_2 = \gamma\left(x_1 - vt_1\right) \qquad \text{and} \qquad x_2' = \gamma\left(x_1' - vt_1'\right)$$

- Relationship between perceived lengths is thus

$$L_2 = x_2' - x_2 = \gamma\Big((x_1' - x_1) - v(t_1' - t_1)\Big) = \gamma L_1 \tag{14}$$

We see length contracted by $L_1 = (1/\gamma)L_2$. This is often written as $\ell' = \ell_0/\gamma$.

Relativistic velocity

Take derivatives of Eqs. 1 through 4 to give

$$\begin{aligned} dx_2 &= \gamma(dx_1 - v\,dt_1) \qquad \text{and} \qquad dy_2 = dy_1 \qquad \text{and} \qquad dz_2 = dz_1 \\ dt_2 &= \gamma(dt_1 - \frac{\beta}{c}\,dx_1). \end{aligned}$$

The velocity in frame 2 is the change in position dx_2 divided by the change in time dt_2. Divide numerator and denominator by dt_1:

$$\begin{aligned} v_{2,x} &= \frac{dx_2}{dt_2} = \frac{\gamma(dx_1 - v dt_1)}{\gamma(dt_1 - (v/c^2) dx_1)} = \frac{(dx_1/dt_1) - v(dt_1/dt_1)}{(dt_1/dt_1) - (v/c^2)(dx_1/dt_1)} \\ &= \frac{v_{1,x} - v}{1 - \dfrac{v\,v_{1,x}}{c^2}} \end{aligned} \tag{15}$$

Relativistic velocity II

Calculate $v_{2,y}$ and $v_{2,z}$ in the same manner:

$$
\begin{aligned}
(16)\qquad v_{2,y} &= \frac{dy_2}{dt_2} = \frac{dy_1}{\gamma\left[dt_1 - (v/c^2)dx_1\right]} = \frac{(dy_1/dt_1)}{\gamma\left[(dt_1/dt_1) - (v/c^2)(dx_1/dt_1)\right]} \\
&= \frac{v_{1,y}}{\gamma\left[1 - \dfrac{v\, v_{1,x}}{c^2}\right]} \\
(17)\qquad v_{2,z} &= \frac{v_{1,z}}{\gamma\left[1 - \dfrac{v\, v_{1,x}}{c^2}\right]}.
\end{aligned}
$$

Relativistic velocity: inverse equations

The inverses of the relativistic velocity expressions are

$$v_{1,x} = \frac{v_{2,x} + v}{1 + \frac{v v_{2,x}}{c^2}} \tag{18}$$

$$v_{1,y} = \frac{v_{2,y}}{\gamma \left[1 + \frac{v v_{2,x}}{c^2}\right]} \tag{19}$$

$$v_{1,z} = \frac{v_{2,z}}{\gamma \left[1 + \frac{v v_{2,x}}{c^2}\right]}. \tag{20}$$

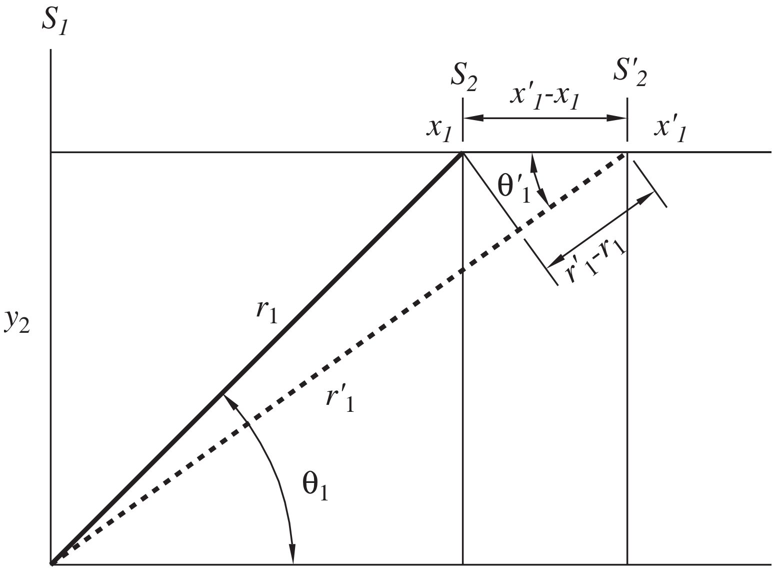
Classical Doppler shift

Consider a source emitting sound at a frequency ν_0. The sound travels in a medium at a velocity c. If source is moving at a velocity v relative to the medium such that $\beta = v/c$, the frequency observed by an observer at rest relative to medium is

$$\nu' = \nu_0 \left(\frac{1}{1 + \beta \cos\theta} \right) \tag{21}$$

Relativistic Doppler shift I

- Observer at rest in S_1. Source in frame S_2 at speed v in $\hat{x}$ direction.
- Emitter is stationary in frame S_2, so $x_2' = x_2$ and we can set both to zero.
- All agree on $y_1 = y_1' = y_2$.
- "Crests" of the electric field are emitted in frame S_2 at the times t_2 and t_2', giving a period of $T_2 = t_2' - t_2 = 1/\nu_2$.
- In frame S_1, the electric field crests are emitted at $t_1 = \gamma t_2$ and $t_1' = \gamma t_2'$.



Relativistic Doppler shift II

- Observer in S_1 also has to wait for crests to reach observation point. This adds in time delays of r_1/c and r_1'/c, respectively.
- Time difference between crests as perceived by observer in S_1 is thus

$$T_1 = (t_1' - t_1) + (\frac{r_1'}{c} - \frac{r_1}{c}) = \gamma(t_2' - t_2) + \frac{r_1' - r_1}{c} \tag{22}$$

or

$$T_1 = \gamma T_2 + (r_1' - r_1)/c. \tag{23}$$

- From Eq. 1, position shift perceived by stationary observer in S_1 is

$$x_1' - x_1 = \gamma\Big((x_2' - x_2) + v(t_2' - t_2)\Big) = \gamma v T_2 \tag{24}$$

since $x_2' = x_2 = 0$.

Relativistic Doppler shift III

- Now $c = \lambda/T$, so $vT_2 = (v/c)\lambda_2 = \beta\lambda_2$. Therefore to frame S_1 the source appears to move by $\gamma\beta\lambda_2$. As we shall see in Eq. 28 the wavelength observed in S_1 is approximately $\lambda_1 = \lambda_2/\gamma$, so even a very relativistic source ($\beta \longrightarrow 1$) appears to move only by a wavelength λ. For a distant observer, $\theta'_1 \simeq \theta_1$.
- Radial distance difference $r'_1 - r_1$ is then

$$r'_1 - r_1 = (x'_1 - x_1)\cos\theta'_1 \simeq \gamma v T_2 \cos\theta_1 . \tag{25}$$

- Now use Eq. 25 in Eq. 23 to give

$$T_1 = \gamma T_2 + (r'_1 - r_1)/c = \gamma T_2 + \gamma\beta T_2 \cos\theta_1 = \gamma T_2[1 + \beta\cos\theta_1]. \tag{26}$$

Relativistic Doppler shift IV

- Take reciprocal to get the frequency:

$$\nu_1 = \frac{\nu_2}{\gamma[1+\beta\cos\theta_1]} \qquad \text{or} \qquad \nu' = \frac{\nu_0}{\gamma(1+\beta\cos\theta)} \tag{27}$$

- Also since $\lambda = cT$, we have

$$\lambda_1 = \gamma\lambda_2[1+\beta\cos\theta_1] \qquad \text{or} \qquad \lambda' = \lambda_0\,\gamma(1+\beta\cos\theta) \tag{28}$$

  or a red-shift in wavelength ($\theta = 0$ corresponds to receding).

- These are the general results for the relativistic Doppler shift. Recall again that the classical Doppler shift (Eq. 21) for an observer at rest with respect to the medium, and a moving source, goes like

$$\nu' = \frac{\nu_0}{1+\beta\cos\theta}$$

  so the difference is a factor of γ. In a vacuum, there's no medium and one can't distinguish the case from the source stationary in the medium or the emitter stationary in the medium.

Relativistic Doppler shift V

- Again, we have

  Relativistic: $\nu' = \dfrac{\nu_0}{\gamma(1 + \beta\cos\theta)}$

  Classical, observer stationary in medium: $\nu' = \dfrac{\nu_0}{1 + \beta\cos\theta}$

  As you can see, the relativistic and classical cases differ by a factor of γ, and of course as $\beta \longrightarrow 0$ then $\gamma \longrightarrow 1$ so the **correspondence principle** is again demonstrated!

- As an example of how the relativistic and classical Doppler shifts differ, consider the case when $\theta = \pi/2$. In the classical Doppler shift, there is no shift; the relativistic result is

$$\nu'_{\text{source perpendicular}} = \frac{\nu_0}{\gamma}.$$

Relativistic Doppler shift VI

- When the source is moving straight towards the observer, we have $\theta_1 = \pi$ and

$$\nu'_{\text{source directly towards}} = \frac{\nu_0}{\gamma(1-\beta)} = \nu_0\sqrt{\frac{1+\beta}{1-\beta}} \simeq \nu_0(1+\beta) \text{ for } \beta \ll 1 \tag{29}$$

- When the source is moving straight away from the observer, we have $\theta_1 = 0$ and

$$\nu'_{\text{source directly away}} = \frac{\nu_0}{\gamma(1+\beta)} = \nu_0\sqrt{\frac{1-\beta}{1+\beta}} \simeq \nu_0(1-\beta) \text{ for } \beta \ll 1 \tag{30}$$

- Attributed to Andrzej Kudlicki, according to `http://home.achilles.net/~ypvsj/humour/jokes.html`:

> **Question:** What's the easiest way to observe Doppler's effect optically (not accoustically) in one's everyday life?
>
> **Answer:** Go out in the evening and look at the cars. Their lights are white or yellow when they approach, but they are red when they are moving away of you.

Hubble constant

- From measuring the relativistic Doppler redshift of common spectroscopic lines (like the hydrogen wavelengths we will learn about when we get to quantum mechanics) we can measure the recessional velocity of a distant star.
- If we think the star is like our sun, we can estimate the distance by comparing the light we receive relative to what we get from the sun.
- Eq. 30 lets you find the velocity from the frequency shift. Edwin Hubble in 1929, summarizing estimates of galaxy redshifts versus distance:

  > The results establish a roughly linear relation between velocities and distances among nebulae for which velocities have been previously published, and the relation appears to dominate the distribution of velocities.

  Read his original paper:

  `http://antwrp.gsfc.nasa.gov/diamond_jubilee/1996/hub_1929.html`

  That is, $v = H_0 x$, and Hubble estimated $H_0 \simeq 500$ km/sec per megaparsec.
- Note: 1 parsec=distance to an object which has a parallax of one arc second as viewed from Earth six months apart=3.261 light years=3.086×10^{16} meters.

Hubble's figure

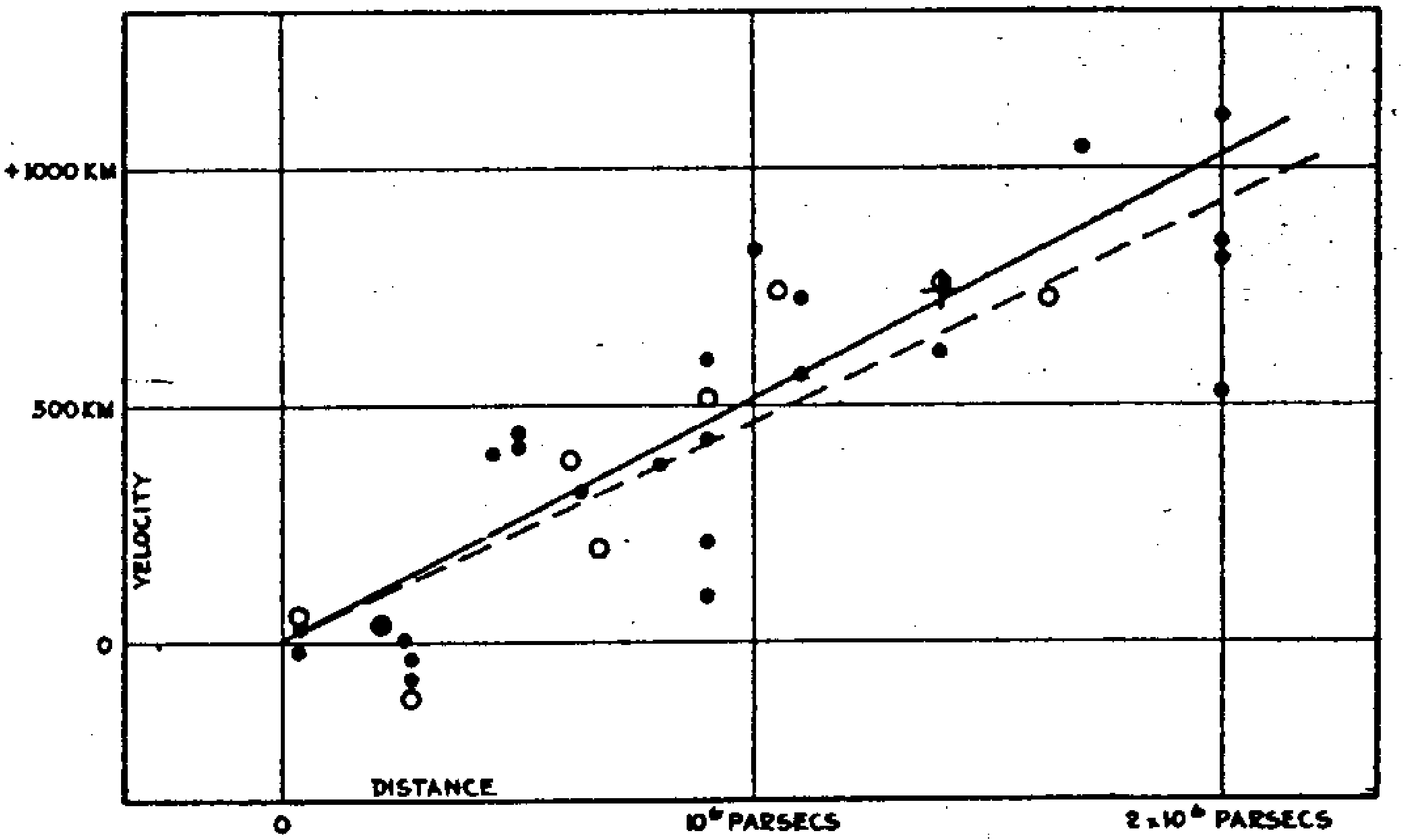


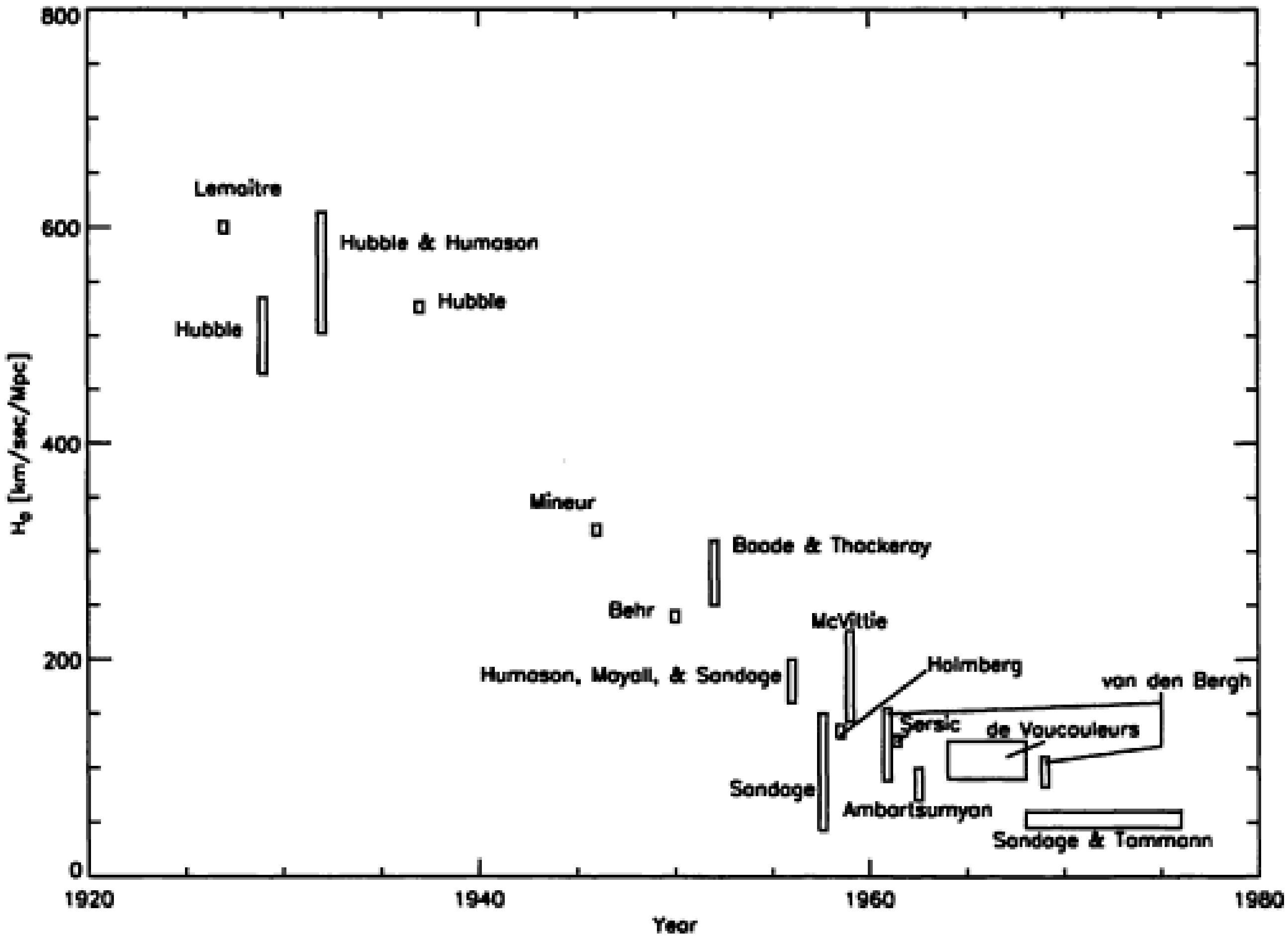
FIGURE 1

Hubble constant II

- Hubble's value of $H_0 \simeq 500$ km/sec/MPc was way off from modern estimates of about 72 km/sec/MPc. One paper that discusses the history of H_0 estimates is here.
- What does it mean? Since $v = \Delta x/\Delta t$ and $v = H_0 x$, we should think of $1/H_0$ as a time, in which case 70 km/sec/Mpc works out to 14 billion years.

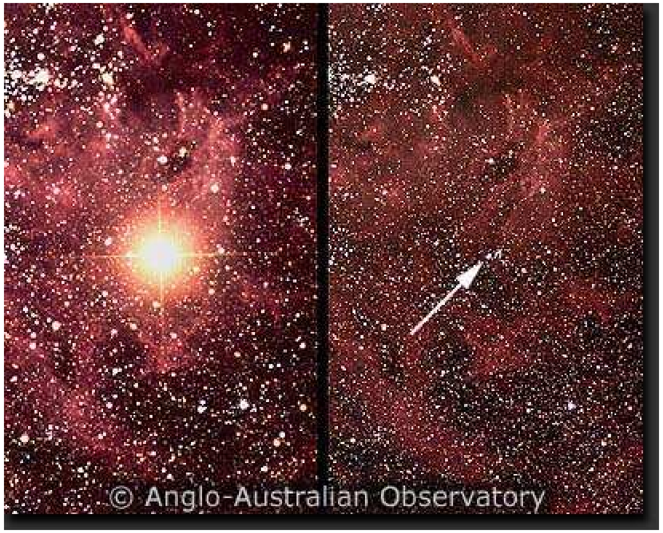
Hubble constant: improving estimates

Here’s the plot of how the measurements have been refined over time; this is from the same paper as before:



Is Hubble's constant constant?

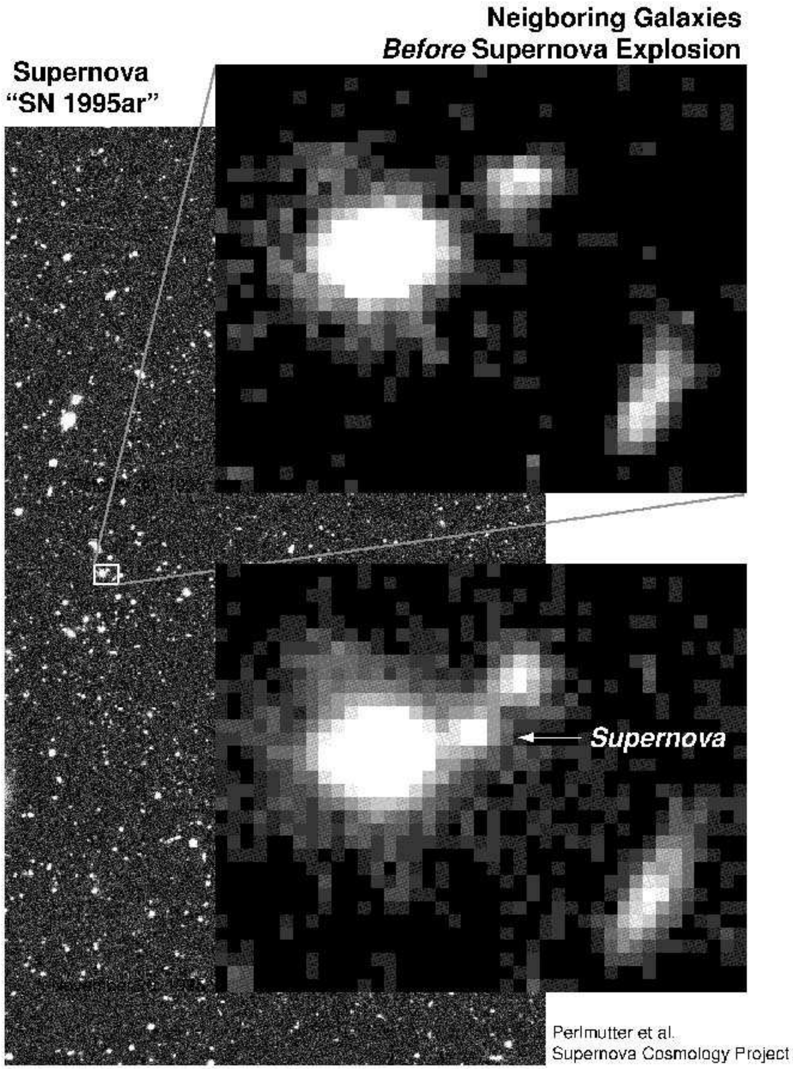
- Has the expansion rate of the universe been constant over time?
- Strategy: use Type Ia supernovæ as "standard candles" since their total radiation power should reach a fairly well defined peak.
- From measuring the amount of light we observe from such a "standard candle" we can calculate its distance.
- Problem: in a typical galaxy ($\sim 10^{11}$ stars) there may only be two or three type Ia supernovae in a thousand years!
- Supernovae that could be seen by the naked eye or even by binoculars: the most recent was in 1987, and the next most recent was 400 years before!
- How can we get many data points?



Supernova SN1987a, after and (from archived telescope images) before. At the time, it was thought rather silly and optimistic to label the first (and of course only) supernova of the year with "a". . .

Hunting for supernovae

- Take lots of pictures of dark regions in the sky, and save them.
- Take pictures of the same regions a week or two later. Use a computer to hunt for differences.
- Now in a few years you can get data on ~ 100 supernovae!



Perlmutter et al.
Supernova Cosmology Project

Is Hubble's constant constant? Maybe not!

See article by Saul Perlmuter, *Physics Today*, April 2003, p. 53 (you can get it here).