Conservation of momentum I

- “Conservation may be a sign of personal virtue, but it is not a sufficient basis for a sound, comprehensive energy policy.” Former Vice President Dick Cheney, April 30, 2001.
- In science, conservation of energy is one of our most fundamental laws, and we can even sort of believe it.
- But what about the conservation of momentum mv? What’s so special about that? Why is it conserved?

Conservation of momentum II

- Well, think of a snapshot of several balls on a pool table. We can calculate the center of mass position from

$$\vec{r}_{\text{cm}} = \frac{\sum \vec{r}_i m_i}{\sum m_i} \tag{1}$$

- If no external force acts on the system, then there's no acceleration on $\vec{r}_{\text{cm}}$ and its velocity remains unchanged.
- If we now shift into the frame moving at the constant velocity of the position $\vec{r}_{\text{cm}}$ of the center of mass position, we have made $\vec{r}_{cm}$ stationary. No external force means no change in velocity; so velocity is zero! $d\vec{r}_{\text{cm}}/dt = 0$!
- Taking d/dt of Eq. 1 then gives $\sum m_i v_i = 0$, or **conservation of momentum**.

Relativistic momentum I

- Conservation of momentum involves constant center of mass with no external forces, and bookkeeping on how this changes between different inertial frames.
- Special relativity tells us to be more careful in shifting between different inertial frames.
- Since momentum involves mass and velocity, let's remember the velocity transforms we've derived before:

$$v_{2,x} = \frac{v_{1,x} - v}{1 - v v_{1,x}/c^2} \quad (2)$$

$$v_{2,y} = \frac{v_{1,y}}{\gamma(1 - v v_{1,x}/c^2)} \quad (3)$$

$$v_{2,z} = \frac{v_{1,z}}{\gamma(1 - v v_{1,x}/c^2)} \quad (4)$$

Relativistic momentum: $\vec{p} \perp \vec{v}$

- Let's pick our frame S_1 to have $v_{1,x} = 0$ and $p_{1,x} = 0$, and ask what observers in frame S_2 (at velocity v relative to S_1) see.
- In the $\hat{y}$ direction (orthogonal to, or $\perp \hat{x}$), we use velocity transformation from Eq. 3 of $v_{2,y} = v_{1,y}/\gamma[1 - v\,v_{1,x}/c^2]$ to give

$$p_{2,y} = m_0 v_{2,y} = m_0 \frac{v_{1,y}}{\gamma(1 - v v_{1,x}/c^2)} = m_0 \frac{v_{1,y}}{\gamma(1 - v \cdot 0/c^2)} = \frac{m_0 v_{1,y}}{\gamma}, \tag{5}$$

giving $p_{1,y} = m_0 v_{1,y} = \gamma p_{2,y}$ or

$$p_{1,y} = \gamma m_0 v_{2,y}. \tag{6}$$

This can be interpreted by saying that the inertial mass m_0 of the particle in frame S_2 looks to us in S_1 as if it has increased by factor γ. "Mass increase"?
- If we brought particle to rest in our frame we would measure the same "rest mass" m_0.
- Same thing happens in $\hat{z}$ direction (again, orthogonal to $\hat{x}$ direction of frame shift velocity v).

Relativistic forces: $\vec{F} \perp \vec{v}$

- Force is defined as the change in momentum per time:

$$\vec{F} \equiv \frac{d(m_0\gamma\vec{v})}{dt} = m_0\gamma\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt} + \gamma\vec{v}\frac{dm_0}{dt} \qquad (7)$$

  The rest mass is the rest mass, so $dm_0/dt = 0$.
- Consider case when force is always perpendicular to velocity (for example, charged particle in a magnetic field). Now $\vec{F} \perp \vec{v}$. No velocity and therefore no motion along $\vec{F}$ direction, so no work!
- Direction of $\vec{v}$ changes but magnitude v does not, so $d\gamma/dt = 0$. We are therefore left with

$$\vec{F} = m_0\gamma\frac{d\vec{v}}{dt} = m_0\gamma\vec{a} \qquad (\text{for } \vec{F} \perp \vec{v}) \qquad (8)$$

Forces $\vec{F} \perp \vec{v}$ continued

- An excellent example of $\vec{F} \perp \vec{v}$ is a centripetal force. An acceleration of $a = v^2/r$ on an object is required (in its own frame) to keep it in uniform circular motion.
- But in relativity, our observations in the lab frame (for example, when we're sitting on the magnet watching the particle circle around) involve $\vec{F} = m_0\gamma\vec{a}$ rather than $\vec{F} = m\vec{a}$.
- Therefore in our reference frame the centripetal force is

$$\vec{F} = m_0\gamma\vec{a} = -m_0\gamma\frac{v^2}{r} \tag{9}$$

- If the centripetal force required to maintain uniform circular motion is provided by a magnetic field B in our frame according to $\vec{F} = q\vec{v} \times \vec{B}$ or $F = qvB$ when $\vec{v} \perp \vec{B}$, we have

$$F_{\text{magnetic}} = F_{\text{centripetal}} \qquad \text{or} \qquad qvB = \frac{m_0\gamma v^2}{r} \tag{10}$$

giving

$$r = \frac{m_0\gamma v}{qB}. \tag{11}$$

Verified experimentally in 1909.

Now consider $\vec{F} \parallel \vec{v}$

- When $\vec{F} \parallel \vec{v}$, the particle's speed and thus γ will *not* be constant. Return to Eq. 7 with $dm_0/dt = 0$:

$$\vec{F} \equiv \frac{d(m_0\gamma\vec{v})}{dt} = m_0\gamma\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt}$$

  Again, $d\vec{v}/dt = \vec{a}$. Calculate $d\gamma/dt$:

$$\frac{d}{dt}(1 - v^2/c^2)^{-1/2} = -\frac{1}{2}(1 - v^2/c^2)^{-3/2}(-2)\frac{\vec{v}}{c^2}\frac{d\vec{v}}{dt} = \gamma^3\frac{\vec{v}}{c^2}\vec{a} \tag{12}$$

  because it's only when we have the square of velocity that we lose information on its direction.
- Returning to Eq. 7 we now have

$$\begin{aligned} \vec{F} &= m_0\gamma\vec{a} + m_0\vec{v}\,\gamma^3\frac{\vec{v}}{c^2}\vec{a} = m_0\gamma\vec{a}\left(1 + \gamma^2\frac{v^2}{c^2}\right) \qquad (13) \\ &= m_0\gamma\vec{a}\left(1 + \frac{v^2}{c^2 - v^2}\right) = m_0\gamma\vec{a}\left(\frac{c^2 - v^2}{c^2 - v^2} + \frac{v^2}{c^2 - v^2}\right) \\ &= m_0\gamma\vec{a}\,\frac{c^2}{c^2 - v^2} = m_0\gamma\vec{a}\,\frac{1}{1 - v^2/c^2} = m_0\gamma^3\vec{a}. \end{aligned}$$

Kinetic energy

- We now know how a relativistic transform for forces along direction of the frame shift velocity, or when $\vec{F} \parallel \vec{v}$.
- So let's calculate the kinetic energy based on the work required to accelerate a particle:

$$E_k = \int_0^{x'} F\,dx = \int_0^{x'} m_0\gamma^3 a\,dx. \tag{14}$$

- Now

$$a\,dx = \frac{dv}{dt}\,dx = \frac{dx}{dt}\,dv = v\,dv, \tag{15}$$

so we can write the kinetic energy as

$$E_k = \int_0^{v'} \gamma^3 m_0 v\,dv = m_0 \int_0^{v'} \frac{v}{(1 - v^2/c^2)^{3/2}}\,dv. \tag{16}$$

Kinetic energy II

- Again we had from Eq. 16

$$E_k = m_0 \int_0^{v'} \frac{v}{(1 - v^2/c^2)^{3/2}}\, dv.$$

Define $A \equiv c^2(1 - v^2/c^2)^{-1/2}$ so that

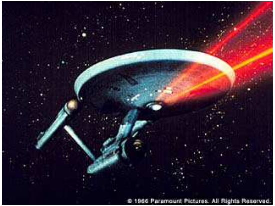
$$dA = c^2(-1/2)(1 - v^2/c^2)^{-3/2}(-2v/c^2)\, dv = \frac{v}{(1 - v^2/c^2)^{3/2}}\, dv. \tag{17}$$

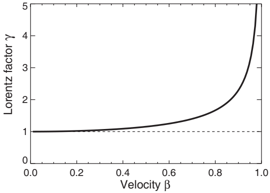
- Therefore, we can recognize Eq. 16 as $\int dA = A$ and obtain

$$\begin{aligned} E_k &= \frac{m_0 c^2}{(1 - v^2/c^2)^{1/2}} \Big|_0^{v'} \qquad (18) \\ &= m_0 c^2 \left(\frac{1}{(1 - v'^2/c^2)^{1/2}} - 1 \right) \\ &= (\gamma - 1) m_0 c^2. \end{aligned}$$

Kinetic energy III

- The relationship of $E_k = (\gamma - 1)m_0c^2$ is why Star Trek is science *fiction*.
- Recall that γ goes screaming up to large values as $v \to c$ or $\beta \to 1$
- We need lots (nearly infinite) energy to even approach the speed of light; and only about 250 stars are as close 10 Parsecs or 33 light years away (see http://www.recons.org; Alpha Centauri is 4.2 ly away).
- We may reach for the stars, but we won't get there! (Unless exotic, unproven physics turns out to work—like wormholes).





Correspondence principle and kinetic energy

- But let's return to earth and consider low velocities.
- In classical limit $v \ll c$, we found found $\gamma \simeq 1 + \frac{1}{2}\frac{v^2}{c^2}$.
- Therefore the classical limit of relativistic kinetic energy is

$$E_k \simeq \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) m_0 c^2 = \frac{1}{2} m_0 v^2 \tag{19}$$

  as expected (**correspondence principle**).
- In the highly relativistic limit of $\gamma \gg 1$, we instead obtain

$$E_k \simeq \gamma m_0 c^2. \tag{20}$$

Particles in motion

- $E_k = (\gamma - 1)m_0c^2$ suggests a new interpretation of the total energy of a particle in motion.
- Assume that a particle at rest has some energy E_0 associated with it:

$$E = E_0 + E_k. \tag{21}$$

- For $\gamma \gg 1$, we found $E \simeq E_k \simeq \gamma m_0 c^2$. Therefore, we make the association $E_0 = m_0c^2$ which allows us to write the total energy as

$$\begin{aligned} E &= E_0 + E_k \\ &= m_0c^2 + (\gamma - 1)m_0c^2 \end{aligned} \tag{22}$$

  or the sum of rest and kinetic energy.
- OK, so we can write the total energy as $E = \gamma m_0 c^2$, but what's really radical here is the energy equivalence of mass:

$$E = m_0c^2 \tag{23}$$

$E = mc^2$

Rest energy and electron-Volts

- $E_0 = m_0 c^2$ is probably the best-known result of modern physics. "Weigh" particles in natural units for atomic and nuclear physics calculations.
- The electron-Volt, or eV: energy gained by an electron as it experiences an electrostatic potential change of one volt. Work is $W = qV$, giving

$$1 \text{ eV} = 1.6 \times 10^{-19} \frac{\text{Coulomb}}{e^- \text{ charge}} \cdot 1 \text{ Volt} = 1.6 \times 10^{-19} \text{ Joule}. \quad (24)$$

- Proton mass in eV:

$$\begin{aligned} E_0 &= m_0 c^2 = 1.67 \times 10^{-27} \text{ kg} \cdot (3 \times 10^8 \text{ m/sec})^2 \quad (25) \\ &= 1.5 \times 10^{-10} \text{ Joules} \cdot \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ Joules}} \\ &= 939 \times 10^6 \text{ eV}. \end{aligned}$$

The mass $m_0 = E_0/c^2$ can then be written as $m_0 = 939 \text{ MeV}/c^2$. Sloppy version: "the proton mass is 939 MeV" or "the electron mass is 511 keV."

Chemical reactions

- Some chemical bond energies: H-C bond 80.9 kcal/mol, C-N bond 184 kcal/mol, C-O 257 kcal/mol.
- Convert H-C to kJ/mol:

$$80.9\ \frac{\text{kcal}}{\text{mol}} \cdot 4.184\ \frac{\text{kJ}}{\text{kcal}} = 338\ \text{kJ/mol}.$$

- Convert to eV/atom:

$$338\ \frac{\text{kJ}}{\text{mol}} \cdot \frac{10^3\ \text{J}}{\text{kJ}} \cdot \frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}} \cdot \frac{1\ \text{mol}}{6.02 \times 10^{23}} = 3.50\ \text{eV}/\text{bond}$$

- Equating to m_0c^2 gives a fractional mass change for electron of

$$\frac{3.50\ \text{eV}}{511 \times 10^3\ \text{eV}} = 7 \times 10^{-6}$$

- Difficult to detect any mass change due to chemical bonding! **Correspondence principle** in action again.