Quantum statistics: review

- Fundamental equation: $n(E) = g(E)\,f(E)\,dE$.
  - Number $n(E)$ of particles is density of available states $g(E)$ times probability of occupying those states $f(E)$.
- Maxwell-Boltzmann: non-interacting, non-integer occupancy of states. Ideal gas.

$$f_{MB}(E) = \frac{1}{\exp[E/k_B T]} \tag{1}$$

- Bose-Einstein: integer occupancy of 0, 1, 2. . .. Photons in a cavity, lasers.

$$f_{BE}(E) = \frac{1}{\exp[E/k_B T] - 1} \tag{2}$$

- Fermi-Dirac: integer occupancy of either 0 or 1. Electrons; Pauli exclusion principle. Fermi energy is $E_F = \frac{h^2}{2m_e}\left(\frac{3N}{8\pi V}\right)^{2/3}$, where N/V is density of valence electrons.

$$f_{FD}(E) = \frac{1}{\exp[(E - E_f)/k_B T] + 1} \tag{3}$$

f_{MB} and f_{BE}

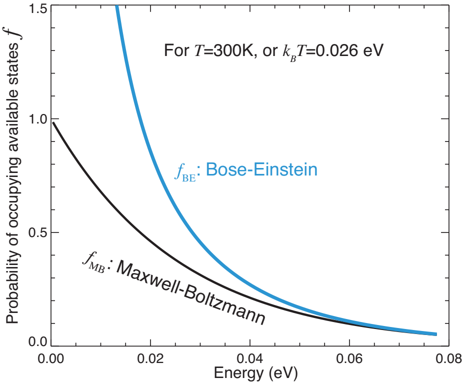
Here's a plot of

$$f_{MB}(E) = \frac{1}{\exp[E/k_B T]}$$

and

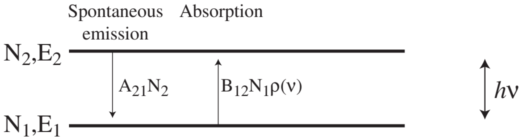
$$f_{BE}(E) = \frac{1}{\exp[E/k_B T] - 1}$$

at room temperature for $T = 300\text{K}$.



Einstein and radiation

- This is done in Serway Sec. 12.7. Consider a two-level system, with energies $E_2 - E_1 = h\nu$, and populations N_1 and N_2:



- **Spontaneous emission**: the rate at which we lose electrons from state N_2 is proportional to the number of electrons in that state:

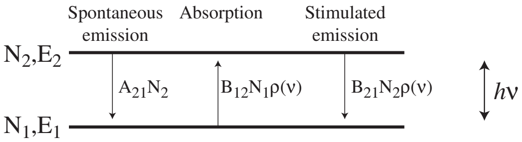
$$\left(\frac{dN_2}{dt}\right)_{\text{spont}} = -A_{21}N_2 \qquad (4)$$

- **Absorption**: the rate at which we pump electrons up to state N_2 is proportional to the number of electrons in state N_1 and the photon density $\rho(\nu)$:

$$\left(\frac{dN_1}{dt}\right)_{\text{abs}} = -\left(\frac{dN_2}{dt}\right)_{\text{abs}} = -B_{12}N_1\rho(\nu) \qquad (5)$$

Einstein and radiation II

- Einstein proposed a third process:



- **Stimulated emission**: we can also drive transitions from state 2 to state 1 in proportion to the population of state 2 and the photon density $\rho(\nu)$:

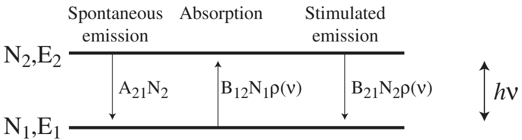
$$\left(\frac{dN_2}{dt}\right)_{\text{stim}} = -B_{21}N_2\rho(\nu) \tag{6}$$

- And remember our other two processes:

$$\left(\frac{dN_2}{dt}\right)_{\text{spont}} = -A_{21}N_2 \qquad \text{and} \qquad \left(\frac{dN_2}{dt}\right)_{\text{abs}} = B_{12}N_1\rho(\nu)$$

Einstein and radiation III

- Assume thermodynamic equillibrium, and assume $\rho(\nu)$ is the Planck blackbody spectrum.



- With the system in equilibrium, N_1 and N_2 evolve towards constant values. As a result,

$$\frac{dN_2}{dt} = 0 = -N_2 A_{21} - N_2 B_{21} \rho(\nu) + N_1 B_{12} \rho(\nu) \tag{7}$$

which gives

$$\begin{aligned}(N_1 B_{12} - N_2 B_{21})\rho(\nu) &= N_2 A_{21} \\ \rho(\nu) &= \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}} = \frac{A_{21}}{B_{12}(N_1/N_2) - B_{21}}\end{aligned}$$

Einstein and radiation IV

- Again,

$$\rho(\nu) = \frac{A_{21}}{B_{12}(N_1/N_2) - B_{21}}$$

- Now use $\rho(\nu)$ as provided by a Planck blackbody spectrum, and realize that $N_1/N_2 = \exp[h\nu/k_B T]$. This gives

$$\begin{aligned}
\frac{8\pi h\nu^3}{c^3}\frac{1}{\exp[h\nu/k_B T] - 1} &= \frac{A_{21}}{B_{12}\exp[h\nu/k_B T] - B_{21}} \\
\frac{8\pi h\nu^3}{c^3}B_{12}\exp[h\nu/k_B T] - \frac{8\pi h\nu^3}{c^3}B_{21} &= A_{21}\exp[h\nu/k_B T] - A_{21} \\
\left(\frac{8\pi h\nu^3}{c^3}B_{12} - A_{21}\right)\exp[h\nu/k_B T] &= \left(\frac{8\pi h\nu^3}{c^3}B_{21} - A_{21}\right) \\
\left(\frac{8\pi h\nu^3}{c^3}\frac{B_{12}}{B_{21}} - \frac{A_{21}}{B_{21}}\right)\exp[h\nu/k_B T] &= \left(\frac{8\pi h\nu^3}{c^3} - \frac{A_{21}}{B_{21}}\right)
\end{aligned}$$

- This must be true for any temperature T! The only way that can be so is for the quantities inside $()$ to be zero on either side of the equation!

Einstein and radiation V

- Pick the right hand term:

$$\left(\frac{8\pi h\nu^3}{c^3} - \frac{A_{21}}{B_{21}}\right) = 0 \qquad \rightarrow \qquad \frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}$$

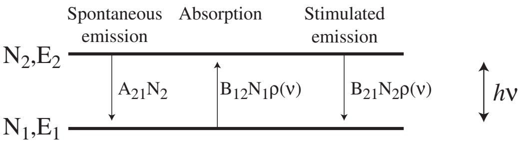
That is, the spontaneous emission coefficient A_{21} divided by the stimulated emission coefficient B_{21} scales like ν^3. Stimulated emission declines like ν^{-3} or λ^3 relative to spontaneous emission, so it's easier to get stimulated emission with microwaves than it is with x rays.

- Now use the above result in the left hand term:

$$\left(\frac{8\pi h\nu^3}{c^3}\frac{B_{12}}{B_{21}} - \frac{A_{21}}{B_{21}}\right) = 0 \rightarrow \frac{8\pi h\nu^3}{c^3}\left(\frac{B_{12}}{B_{21}} - 1\right) = 0 \rightarrow B_{12} = B_{21}$$

That is, the stimulated emission and absorption coefficients are one and the same! Recall Fermi's golden rule for transition rates: the rate is the same for $1 \rightarrow 2$ as for $2 \rightarrow 1$.

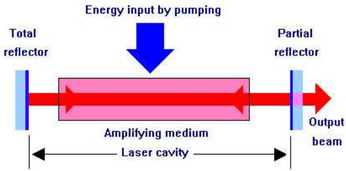
Einstein, atoms, and radiation



- We have $B_{12} = B_{21}$: the stimulated emission and absorption coefficients are one and the same. Calculate via Fermi's golden rule.
- We have $\dfrac{A_{21}}{B_{21}} = \dfrac{8\pi h\nu^3}{c^3}$ for the ratio between spontaneous emission A_{21} and stimulated emission B_{21}.
- Our processes then become $\left(\frac{dN_2}{dt}\right)_{\text{stim}} = -BN_2\rho(\nu)$, $\left(\frac{dN_2}{dt}\right)_{\text{spont}} = -AN_2$, and $\left(\frac{dN_2}{dt}\right)_{\text{abs}} = +BN_1\rho(\nu)$.
- If we can put lots of atoms into state N_2 and have high photon density $\rho(\nu)$, we can have stimulated emission dominate. Since the electric field of one photon stimulates the emission of another, they are in phase with each other.

Lasers

Who invented the laser? Many people were in the stew. Wikipedia has a good, concise history.



Laser cavity

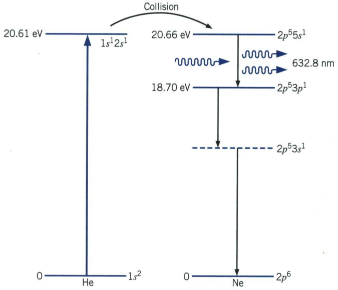
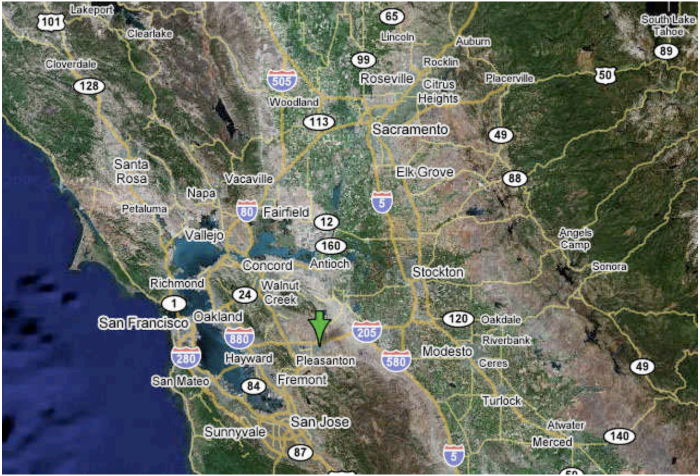


FIGURE 8.20 Sequence of transitions in a He-Ne laser.

Helium-Neon laser scheme (Krane Fig. 8.20)

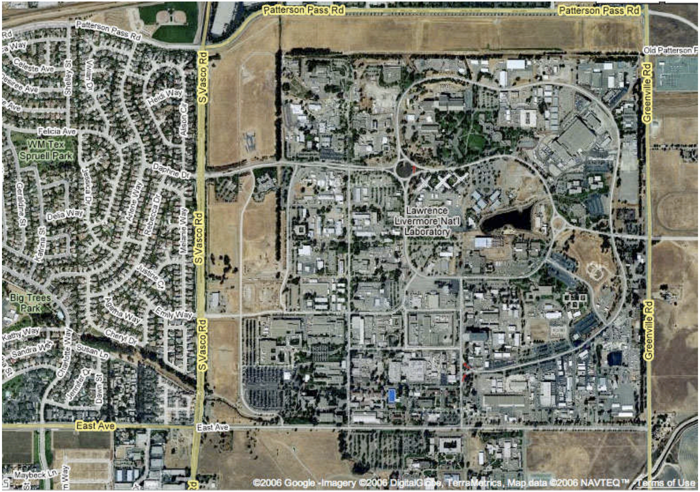
Let's go to Livermore

For the biggest, baddest laser around, let's go to Livermore; it's one of the two nuclear weapons physics lab, along with Los Alamos in New Mexico.



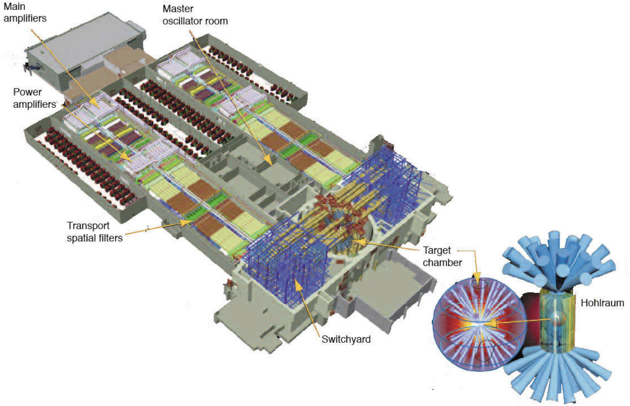
Livermore lab

Lawrence Livermore National Lab. NIF is at the upper right.



NIF: National Ignition Facility

Anticipated operation: 2009. Cost: $1B? Web site:
http://www.llnl.gov/nif/
192 beams (3072 slab amplifiers), total energy of 1.8 MJ, pulse duration 3–20 nsec. During those 3-20 nsec, the lasers emit a power of 5×10^{14} Watts. US electric power generating capacity: 1×10^{12} Watts.



NIF: an aerial view

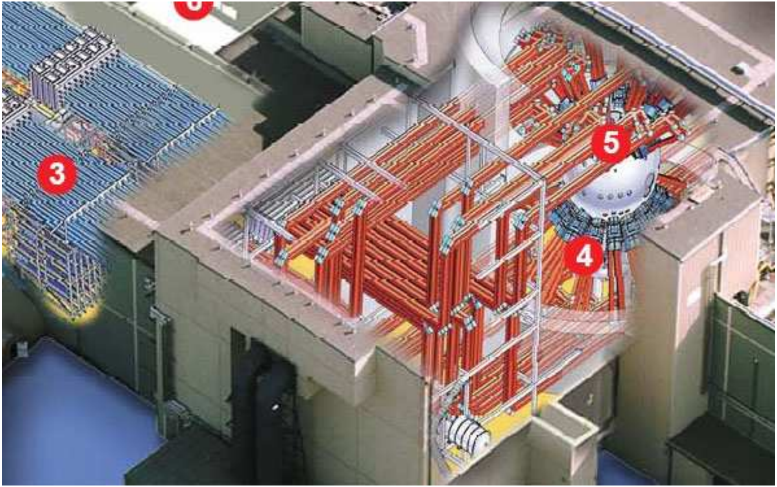
About the size of a football stadium:

NIF components

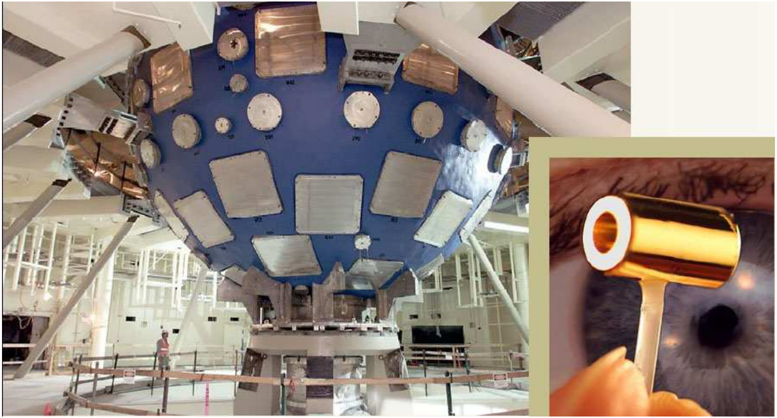
Part of one capacitor bank

One replaceable amplifier slab on its mounting robot

NIF target chamber

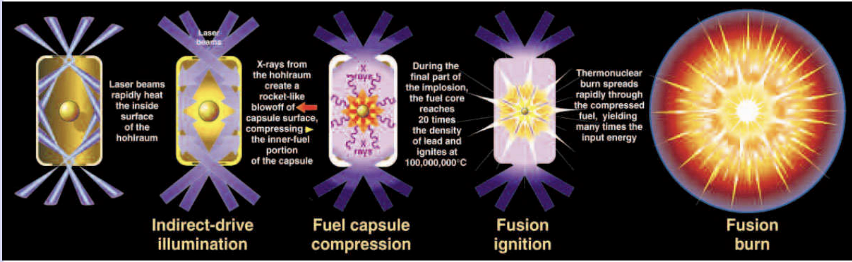


NIF target chamber II, and Hohlraum



Hohlraum leading to fusion

Direct laser heating of pellet produced too many nonuniformities. Indirect heating instead!



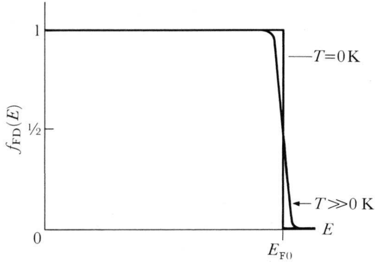
Mini H-bomb. Relevant to understanding weapons, supernovae. Future energy source???

Back to Fermions

If the occupancy of a state can be only 0 or 1 (such as with electrons), we use the already-normalized Fermi-Dirac distribution function (Serway Eq. 10.25):

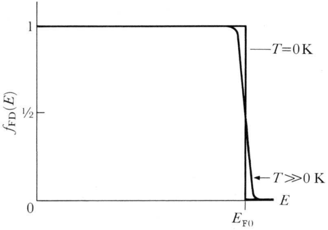
$$f_{\rm FD}(E) = \frac{1}{\exp[(E - E_F)/k_B T] + 1} \qquad (8)$$

Note that $f_{\rm FD}(E) = 1/2$ when $E = E_F$, and that the Fermi energy E_F stands in for a chemical potential (binding of most electrons to particular atoms). Here's Fig. 24-3 of Sandin, *Essentials of Modern Physics*, which is just like Serway Fig. 10.11:



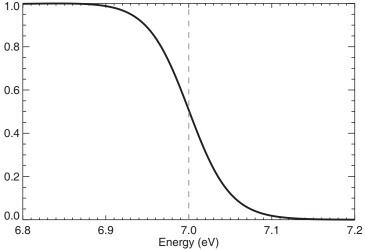
Swimming in the Fermi sea

- Let's populate electrons onto nuclei starting from low energy. Because $f_{\mathrm{FD}}(E)$ stays near 1 and then makes a fairly sharp transition to 0 at $E = E_F$, we will fill up all states till we reach the Fermi energy E_F at which point we'll stop.
- We therefore speak of a *Fermi sea*: there are electrons below the surface of the sea (at E_F), but not above.
- Well, the distribution is not an absolute $1 \longrightarrow 0$ except at zero temperature, so there are some electrons above the Fermi sea. The occupancy of electrons in these energy states will be very low so one electron will rarely encounter another. That is, they appear like non-interacting particles so they are sometimes called a *Fermi gas*.



Floating above the Fermi sea

- In fact the Fermi-Dirac distribution does not make a sudden transition from 1 to 0 at E_F, especially at higher temperatures. Here's a normalized plot of $f_{\mathrm{FD}}(E)$ with $E_F = 7.0$ eV at room temperature:



  What fraction of electrons are above the surface of the Fermi sea?
- A calculation of how much energy it takes to move electrons from the triangle at upper left to the triangle at lower right gives a calculation of the heat capacity of metals of

$$C = \frac{dU}{dT} = 2\frac{Nk_B^2}{E_F}T$$

Fermi sea: conclusion

- Again, $C \propto T$ for electrons, so the heat capacity goes to zero as the temperature goes to zero.
- In other words, when we add energy into the system, we very quickly knock some electrons out from the Fermi sea, and once they're out there are a large number of states available to them.
- Since temperature is the inverse of the log of the number of states made available per energy added, we cannot add much heat into the system without quickly affecting its temperature.
- This has consequences for phenomena including superconductivity.