

PHY 251 Fall 2009: homework problem set 9, due in the PHY 251 drop box in room A-129 by noon on Friday, Nov. 20.

1. Serway 10.2

Answer: From Serway Eq. 10.8, we know that the Maxwell-Boltzmann velocity distribution is

$$n(v) dv = \frac{4\pi N}{V} \left(\frac{m}{2\pi k_B T} \right)^{3/2} v^2 \exp\left[-\frac{mv^2}{2k_B T}\right] dv = B v^2 e^{-Av^2} dv$$

with

$$A \equiv \frac{m}{2k_B T} \quad \text{and} \quad B \equiv \frac{4\pi N}{V} \left(\frac{A}{\pi} \right)^{3/2} = \frac{4N}{\sqrt{\pi} V} A^{3/2}.$$

The most probable speed v_{mp} is at the top of the curve, so we take the derivative and set to zero:

$$\begin{aligned} \frac{d}{dv} B v^2 e^{-Av^2} dv &= B \left(2v e^{-Av^2} + (-2Av)v^2 e^{-Av^2} \right) = 2v B e^{-Av^2} (1 - Av^2) = 0 \\ 1 &= Av^2 \\ v_{\text{mp}} &= \sqrt{\frac{1}{A}} = \sqrt{\frac{2k_B T}{m}} \end{aligned}$$

2. Serway 10.8. Show the derivations.

Answer: We need to rewrite the Maxwell-Boltzmann distribution in terms of kinetic energy K instead of velocity v :

$$K = \frac{1}{2}mv^2 \quad \rightarrow \quad dK = mv dv = \sqrt{2mK} dv \quad \rightarrow \quad \frac{1}{\sqrt{2mK}} dK = dv$$

If we substitute these relationships into the expression for $n(v) dv$ we get

$$\begin{aligned} n(K) dK &= \frac{4\pi N}{V} \frac{m^{3/2}}{2^{3/2}(\pi k_B T)^{3/2}} \frac{2}{m} K \exp\left[-\frac{K}{k_B T}\right] \frac{1}{2^{1/2}m^{1/2}\sqrt{K}} dK \\ &= \frac{2\pi N}{V} \frac{1}{(\pi k_B T)^{3/2}} \sqrt{K} \exp\left[-\frac{K}{k_B T}\right] dK \\ &= A\sqrt{K} \exp\left[-\frac{K}{k_B T}\right] dK \quad \text{with} \quad A \equiv \frac{2\pi N}{V(\pi k_B T)^{3/2}}. \end{aligned}$$

The peak or $K_{\text{most probable}}$ is found by setting the derivative to zero:

$$\begin{aligned} \frac{d}{dK} A\sqrt{K} \exp\left[-\frac{K}{k_B T}\right] &= 0 \\ A \left(\frac{1}{2} \frac{1}{\sqrt{K}} \exp\left[-\frac{K}{k_B T}\right] + \sqrt{K} \exp\left[-\frac{K}{k_B T}\right] \left(\frac{-1}{k_B T} \right) \right) &= 0 \\ \frac{1}{2\sqrt{K}} &= \frac{\sqrt{K}}{k_B T} \\ K_{\text{most probable}} &= \frac{k_B T}{2} \end{aligned}$$

The mean kinetic energy is found from $\langle K \rangle = \int K P(K) dK$. Now $n(K)$ gives the number of particles per volume, so dividing by the number of particles per volume gives the probability $P(K)$. Now let's define $x \equiv K/k_B T$ so $K = x k_B T$ and $dK = k_B T dx$:

$$\begin{aligned} \langle K \rangle &= \int_{K=0}^{\infty} K \frac{1}{N/V} \frac{2\pi N}{V \pi^{3/2} (k_B T)^{3/2}} \sqrt{K} \exp\left[-\frac{K}{k_B T}\right] dK \\ &= \int_{x=0}^{\infty} \frac{2\pi}{\pi^{3/2} (k_B T)^{3/2}} (x k_B T)^{3/2} \exp[-x] k_B T dx \\ &= \frac{2}{\sqrt{\pi}} (k_B T) \int_{x=0}^{\infty} x^{3/2} \exp[-x] dx = \frac{2}{\sqrt{\pi}} (k_B T) \frac{3}{4} \sqrt{\pi} \\ &= \frac{3}{2} k_B T. \end{aligned}$$

The root mean square kinetic energy is $\sqrt{\langle K^2 \rangle}$. Let's first find $\langle K^2 \rangle$:

$$\begin{aligned} \langle K^2 \rangle &= \int_{K=0}^{\infty} K^2 \frac{1}{N/V} \frac{2\pi N}{V \pi^{3/2} (k_B T)^{3/2}} \sqrt{K} \exp\left[-\frac{K}{k_B T}\right] dK \\ &= \int_{x=0}^{\infty} \frac{2\pi}{\pi^{3/2} (k_B T)^{3/2}} (x k_B T)^{5/2} \exp[-x] k_B T dx \\ &= \frac{2}{\sqrt{\pi}} (k_B T)^2 \int_{x=0}^{\infty} x^{5/2} \exp[-x] dx = \frac{2}{\sqrt{\pi}} (k_B T)^2 \frac{15}{8} \sqrt{\pi} \\ &= \frac{15}{4} (k_B T)^2 \end{aligned}$$

giving

$$K_{\text{rms}} = \sqrt{\langle K^2 \rangle} = \frac{\sqrt{15}}{2} k_B T.$$

3. Serway 10.9

Answer: The mass of an iron atom is

$$(55.8 \text{ amu}) \cdot (1.661 \times 10^{-27} \text{ kg/amu}) = 9.27 \times 10^{-26} \text{ kg}.$$

so we have

$$\bar{v} = \sqrt{\frac{8k_B T}{\pi m}} = \sqrt{\frac{8 \cdot 1.381 \times 10^{-23} \cdot 6000}{\pi \cdot 9.27 \times 10^{-26}}} = 1510 \text{ m/s}$$

which is beyond the speed of sound at standard temperature and pressure (300 m/s) but nowhere near relativistic. We can therefore use the low β expansion of the Doppler shift for light waves:

$$\begin{aligned} f &= f_0 \frac{\sqrt{1 \pm \beta}}{\sqrt{1 \mp \beta}} = f_0 (1 \pm \beta)^{1/2} (1 \mp \beta)^{-1/2} \\ &\simeq f_0 \left(1 \pm \frac{1}{2}\beta\right) \left(1 \pm \frac{1}{2}\beta\right) \simeq f_0 (1 \pm \beta) \\ \Delta f &= f - f_0 = f_0 (1 \pm \beta) - f_0 = \pm \beta f_0 \end{aligned}$$

The fractional Doppler shift is then

$$\frac{\Delta f}{f_0} = \pm\beta = \pm\frac{\bar{v}}{c} = \pm\frac{1510}{2.99 \times 10^8} = \pm 5.05 \times 10^{-6}$$

4. Serway 10.14

Answer: To find the average energy of an electron, we realize that $n(E) dE$ measures the distribution of the number of electrons per volume as a function of energy. Now from Eq. 10.39 we have $g(E) dE = D E^{1/2} dE$ with D a constant defined in a messy way in Eq. 10.40. But let's instead find a way to express D in terms of the Fermi energy E_F by following Serway Eq. 10.43 to find that

$$D = \frac{3N}{2V} \frac{1}{E_F^{3/2}}.$$

Furthermore, the probability of electrons per energy $P(E) dE$ is found from dividing the number density distribution by the number, or

$$P(E) dE = \frac{n(E) dE}{N/V} = \frac{g(E) f_{FD}(E) dE}{N/V}.$$

We can then use $\langle E \rangle = \int E P(E) dE$. Now in arriving at Eq. 10.43 one uses the fact that the Fermi-Dirac distribution $f_{FD}(E)$ is 1 up to the Fermi energy E_F and 0 thereafter, so we can carry out the above integral with $f_{FD}(E) \rightarrow 1$ and change the upper integration limit to E_F and obtain

$$\begin{aligned} \langle E \rangle &= \frac{1}{N/V} \int_0^{E_F} E g(E) dE = \frac{1}{N/V} \int_0^{E_F} E dE^{1/2} dE = \frac{1}{N/V} \frac{2}{5} D E_F^{5/2} \\ &= \frac{1}{N/V} \frac{2}{5} \frac{3}{2} (N/V) \frac{1}{E_F^{3/2}} E_F^{5/2} = \frac{3}{5} E_F. \end{aligned}$$

OK, with that result in hand, we find that the average energy of a conduction electron in copper is

$$\langle E \rangle = \frac{3}{5} E_F = \frac{3}{5} 7.05 \text{ eV} = 4.23 \text{ eV}.$$

Note that this does not depend on temperature at all! Electrons in metals have kinetic energy even at zero temperature. Next, the average energy of molecules in an ideal gas is given by $\frac{3}{2} k_B T$, so the equivalent temperature is

$$\frac{3}{5} E_F = \frac{3}{2} k_B T \quad \Rightarrow \quad T = \frac{2 E_F}{5 k_B} = \frac{2}{5} \frac{7.05 \text{ eV}}{8.62 \times 10^{-5} \text{ eV/K}} = 32,700 \text{ K}$$

Again, the electrons are surprisingly energetic!

5. Serway 10.15

Answer: From

$$E_F = \frac{h^2}{8m} \left(\frac{3N}{\pi V} \right)^{2/3}$$

we can find

$$\begin{aligned}\frac{N}{V} &= \frac{\pi}{3} \left(\frac{8mE_F}{h^2} \right)^{3/2} = \frac{\pi}{3} \left(\frac{8mc^2 E_F}{(hc)^2} \right)^{3/2} \\ &= \frac{\pi}{3} \left(\frac{8 \cdot 511 \times 10^3 \cdot 11.63 \text{ eV}^2}{(1240 \text{ eV} \cdot \text{nm})^2} \right)^{3/2} = 180 \frac{\text{electrons}}{\text{nm}^3} \left(\frac{10^7 \text{ nm}}{\text{cm}} \right)^3 = 1.8 \times 10^{23} \frac{\text{electrons}}{\text{cm}^3}\end{aligned}$$

The number of atoms per unit volume is

$$\frac{\rho \cdot N_A}{A} = \frac{(2.70 \text{ g/cm}^3) \cdot (6.02 \times 10^{23} \text{ atoms/mol})}{27.0 \text{ g/mol}} = 6.02 \times 10^{22} \text{ atoms/cm}^3$$

so the number of valence electrons per aluminum atom is $(18 \cdot 10^{22}) / (6.0 \times 10^{22}) = 3.0$.

6. Serway 10.19

Answer: We wish to find E such that $f(E) = 0.99 = a$ when $T = 300 \text{ K}$, or

$$\begin{aligned}a &= \frac{1}{\exp[(E - E_F)/k_B T] + 1} \\ \frac{1}{a} &= \exp[(E - E_F)/k_B T] + 1 \\ -\log a &= \frac{E - E_F}{k_B T} \\ E - E_F &= -k_B T \log a = -(8.617 \times 10^{-5} \text{ eV/K}) \cdot (300 \text{ K}) \cdot \log(0.99) = 0.00026 \text{ eV}\end{aligned}$$

so E is just 0.26 meV below E_F .

7. Serway 10.23

Answer: The volume is $V = (10^{-3} \text{ m})^3 = 10^{-9} \text{ m}^3$. From Table 10.1 on p. 359, we see that the Fermi energy of gold is 5.53 eV, and the electron concentration is 5.90×10^{28} per m^3 . Since 4.000 eV is well below the Fermi energy, we have $f_{\text{FD}}(4.000 \leq E \text{ (eV)} \leq 4.025) = 1$. Also, let's define $E_0 = 4.000 \text{ eV}$ and $\Delta E = 0.025 \text{ eV}$. Thus to calculate the number of electrons from $n(E)dE = g(E)f(E)dE$ we just need to know the density of available states $g(E)$ which is given in Serway Eqs. 10.39 and 10.40:

$$\begin{aligned}\frac{N}{V} &= \int_{E_0}^{E_0+\Delta E} n(E) dE = \int_{E_0}^{E_0+\Delta E} g(E) f_{\text{FD}}(E) dE \\ &\simeq \int_{E_0}^{E_0+\Delta E} g(E) \cdot 1 dE = \int_{E_0}^{E_0+\Delta E} D E^{1/2} dE \quad \text{with} \quad D = \frac{8\sqrt{2}\pi m_e^{3/2}}{h^3}.\end{aligned}$$

We then have

$$\begin{aligned}N &= VD \int_{E_0}^{E_0+\Delta E} E^{1/2} dE = VD \frac{2}{3} E^{3/2} \Big|_{E_0}^{E_0+\Delta E} = VD \frac{2}{3} [(E_0 + \Delta E)^{3/2} - E_0^{3/2}] \\ &\simeq VD \frac{2}{3} E_0^{3/2} \left[\left(1 + \frac{\Delta E}{E_0}\right)^{3/2} - 1 \right] = VD \frac{2}{3} E_0^{3/2} \frac{3}{2} \frac{\Delta E}{E_0} = V D E_0^{1/2} \Delta E.\end{aligned}$$

Maybe it was silly to do the integral with a Taylor series expansion on the result; we could have just used $E^{1/2} \Delta E$ from the outset. . . Anyway, we're now ready to get a numerical result where we'll use mks units:

$$\begin{aligned}
 N &= VDE_0^{1/2} \Delta E = V \frac{8\sqrt{2}\pi m_e^{3/2}}{h^3} E_0^{1/2} \Delta E \\
 &= (10^{-9} \text{ m}^3) \frac{8\sqrt{2}\pi(9.11 \times 10^{-31})^{3/2}}{(6.63 \times 10^{-34})^3} (4.000 \text{ eV} \cdot 1.602 \times 10^{-19} \text{ J/eV})^{1/2} \\
 &\quad (0.025 \text{ eV} \cdot 1.602 \times 10^{-19} \text{ J/eV}) \\
 &= 3.4 \times 10^{17} \text{ electrons}
 \end{aligned}$$

8. Serway 12.5

Answer: Let's start adding up the potential from nearest neighbor first, using $k = 1/4\pi\epsilon_0$, for a positive ion in the chain:

$$\begin{aligned}
 U &= -\frac{ke^2}{r} - \frac{ke^2}{r} + \frac{ke^2}{2r} + \frac{ke^2}{2r} - \frac{ke^2}{3r} - \frac{ke^2}{3r} + \frac{ke^2}{4r} + \frac{ke^2}{4r} + \dots \\
 &= -2\frac{ke^2}{r} \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \right].
 \end{aligned}$$

Now the series expansion of $\ln(1+x)$ is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

so we see that the numerical series is the same as $\ln(1+x)$ with $x = 1$, or $\ln(2)$. Thus we have

$$U = -2\frac{ke^2}{r} \ln(2) = -k\alpha \frac{e^2}{r} \quad \text{with} \quad \alpha = 2 \ln(2).$$

9. Serway 12.15

Answer: The photon frequency is found from $E = h\nu = 1.14 \text{ eV}$, giving

$$\nu = \frac{E}{h} = \frac{1.14 \text{ eV}}{4.136 \times 10^{-15} \text{ eV} \cdot \text{sec}} = 2.76 \times 10^{14} \text{ Hz.}$$

The wavelength is

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.14 \text{ eV}} = 1090 \text{ nm.}$$

10. Serway 12.16

Answer: The longest wavelength is $1.85 \mu\text{m}$ which corresponds to a minimum photon energy of

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{1850 \text{ nm}} = 0.67 \text{ eV.}$$

This is the energy gap between bands.

11. Serway 12.18

Answer: The idea here is to replace Z with Z/κ and use $Z = 1$ because of screening of the nuclear charge by all the bound electrons. Thus for $n = 1$ we have

$$E = (-13.6 \text{ eV})Z^2 \Rightarrow (-13.6 \text{ eV})\left(\frac{1}{\kappa}\right)^2$$

so for Si we have $13.6/12^2=0.094 \text{ eV}$ and for Ge we have $13.6/16^2=0.053 \text{ eV}$ as the effective binding energies for the last two electrons. This gets pretty close to thermal energies of $k_B T = 0.025 \text{ eV}$ at room temperature. We also have a modified Bohr radius of

$$r_1 = \frac{a_0}{Z} \Rightarrow \frac{a_0}{1} \kappa$$

which for Si gives $(0.053 \text{ nm}) \cdot 12 = 0.64 \text{ nm}$ and for Ge gives $(0.053 \text{ nm}) \cdot 16 = 0.85 \text{ nm}$. These last two electrons are delocalized from one atom because nearest neighbor distances are 0.23 nm for Si and 0.24 nm for Ge.