PHY 251 Fall 2009: homework problem set 7, due in the PHY 251 drop box in room A-129 by noon on Friday, Oct. 30.

1. Serway 6.15

   Answer:

   (a) The potential energy from the Coulomb force is $U = \frac{q_1 q_2}{4\pi\epsilon_0 r}$. The $q_1 q_2$ terms are all going to have an electron charge times either -1 (electron) or +1 (proton), and the distances r are going to be $1d$, $2d$, or $3d$. We can then work out the potential seen by the particles from left to right, being careful not to double-count (that is, don't count again the potential of a particle to the left of whatever one we're now considering):

$$\begin{aligned}\Delta U_1 &= \frac{e^2}{4\pi\epsilon_0 d}\left[\frac{(-1)(+1)}{1} + \frac{(-1)(-1)}{2} + \frac{(-1)(+1)}{3}\right] = \frac{e^2}{4\pi\epsilon_0 d}\left[-1 + \frac{1}{2} - \frac{1}{3}\right] \\ \Delta U_2 &= \frac{e^2}{4\pi\epsilon_0 d}\left[\frac{(+1)(-1)}{1} + \frac{(+1)(+1)}{2}\right] = \frac{e^2}{4\pi\epsilon_0 d}\left[-1 + \frac{1}{2}\right] \\ \Delta U_3 &= -\frac{e^2}{4\pi\epsilon_0 d}\left[\frac{(-1)(+1)}{1}\right] = \frac{e^2}{4\pi\epsilon_0 d}\left[-1\right]\end{aligned}$$

   so that the net potential energy is

$$\begin{aligned}U &= \Delta U_1 + \Delta U_2 + \Delta U_3 = \frac{e^2}{4\pi\epsilon_0 d}\left[-1 + \frac{1}{2} - \frac{1}{3} - 1 + \frac{1}{2} - 1\right] \\ &= \frac{e^2}{4\pi\epsilon_0 d}\left[-\frac{6}{6} + \frac{3}{6} - \frac{2}{6} - \frac{6}{6} + \frac{3}{6} - \frac{6}{6}\right] = \frac{e^2}{4\pi\epsilon_0 d}\left[-\frac{14}{6}\right] = -\frac{e^2}{4\pi\epsilon_0 d}\frac{7}{3} = -\frac{7e^2}{12\pi\epsilon_0 d}.\end{aligned}$$

   (b) If we say that the electrons are restricted to a 1D box of width $3d$, that's like saying $L = 3d$ in the infinite quantum well solution, and $n = 1$ for the lowest energy quantum state. Therefore the kinetic energy of allowed electron wavefunctions in the box is

$$E_k = E_1 = 1^2 \frac{\pi^2\hbar^2}{2m(3d)^2} = \frac{h^2}{72md^2}$$

   which should be doubled for two electrons.

   (c) The total energy is $U + E_k$; to find the value of d which minimizes the total energy we set the derivative with respect to d to be zero:

$$\begin{aligned}\frac{\partial}{\partial d}(U + E_k) = 0 &= \frac{\partial}{\partial d}\left(-\frac{7e^2}{12\pi\epsilon_0 d} + \frac{h^2}{36md^2}\right) = \left(-(-1)\frac{7e^2}{12\pi\epsilon_0 d^2} + (-2)\frac{h^2}{36md^3}\right) \\ \frac{7e^2}{12\pi\epsilon_0 d^2} &= \frac{h^2}{18md^3} \\ d &= \frac{4\pi\epsilon_0 h^2}{42me^2} = \frac{4\pi(8.85\times10^{-12})\cdot(6.63\times10^{-34})^2}{42\cdot(9.11\times10^{-31})\cdot(1.602\times10^{-19})^2} \\ &= 4.98\times10^{-11}\text{ meters} = 0.0498\text{ nm}.\end{aligned}$$

   (d) The volume per atom of lithium is found from volume=mass/density, and if each atom occupies a cubic space of length x such that volume=x^3 we have

$$x = \left(\frac{m}{\rho}\right)^{1/3} = \left(\frac{7\text{ g/mole}}{(6.02\times10^{23}\text{ atoms/mole})\cdot(0.53\text{ g/cm}^3)\cdot(10^2\text{ cm/m})^3}\right)^{1/3}$$

or $x = 2.80 \times 10^{-10}$ meters or 0.28 nm. This is longer than $2d = 0.10$ nm but at least in the right ballpark. . .

2. Serway 6.16
Answer: The probability p of being in the region between x_1 and x_2 is given by

$$\begin{aligned} p &= \int_{x_1}^{x_2} |\psi(x)^2|\, dx = \int_{x_1}^{x_2} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right]^2 dx \\ &= \frac{2}{L}\frac{1}{k}\int_{x_1}^{x_2} \sin^2(kx)\, d(kx) \qquad \text{with} \qquad k \equiv \frac{n\pi}{L} \\ &= \frac{2}{kL}\left[\frac{1}{2}kx - \frac{1}{4}\sin(2kx)\right]\Big|_{x_1}^{x_2} \\ &= \frac{2}{n\pi}\left[\frac{n\pi}{2L}(x_2 - x_1) - \frac{1}{4}(\sin(\frac{2n\pi x_2}{L}) - \sin(\frac{2n\pi x_1}{L}))\right] \end{aligned}$$

With a well of width 0.300 nm, and $n = 1$ (ground state), the probability of being within 0.100 nm of the left-hand wall corresponds to $x_2 = 0.100$, $x_1 = 0$, and $L = 0.300$. (Note that because we are always dealing with ratios of x/L we don't need to worry about the exact units). In this case the probability is

$$p = \frac{2}{\pi}\left[\frac{\pi}{2}\frac{0.1 - 0.0}{0.3} - \frac{1}{4}(\sin(\frac{2\pi \cdot 0.1}{0.3}) - \sin(0))\right] = 0.196$$

If we instead use $n = 100$ we find

$$p = \frac{2}{100\pi}\left[\frac{100\pi}{2}\frac{0.1 - 0.0}{0.3} - \frac{1}{4}(\sin(\frac{200\pi \cdot 0.1}{0.3}) - \sin(0))\right] = 0.332$$

which approaches the classical limit of having the particle in this region (1/3)=0.333 of the time.

3. An electron is in a finite square well of width $L = 0.5$ nm and $U = 10.0$ eV. Calculate the characteristic tunneling distance δ, and calculate the energy of the ground state both in the approximation that $U \to \infty$ and with $U = 10.0$ eV.
Answer: For the infinite square well, we have

$$E_n = n^2 \frac{\pi^2\hbar^2}{2mL^2} = n^2 \frac{(hc)^2}{8mc^2L^2} = n^2 \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8 \cdot (511 \times 10^3 \text{ eV}) \cdot (0.5 \text{ nm})^2} = n^2(1.50 \text{ eV})$$

so of course $E_1 = 1.50$ eV. For $U = 10.0$ eV, we have

$$\begin{aligned} \delta &= \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U-E)}} = \frac{hc}{2\pi\sqrt{2mc^2(U-E)}} \\ &= \frac{1240 \text{ eV} \cdot \text{nm}}{2\pi\sqrt{2 \cdot (511 \times 10^3 \text{ eV}) \cdot (10.0 - 1.5 \text{ eV})}} = 0.067 \text{ nm} \end{aligned}$$

The approximate result for the energy in the finite quantum well is then

$$E_n \simeq n^2 \frac{(hc)^2}{8mc^2(L+2\delta)^2} = n^2 \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8 \cdot (511 \times 10^3 \text{ eV}) \cdot (0.5 + 2 \cdot 0.067 \text{ nm})^2} = n^2(0.94 \text{ eV})$$

so the ground state energy is reduced from 1.50 eV to 0.94 eV.

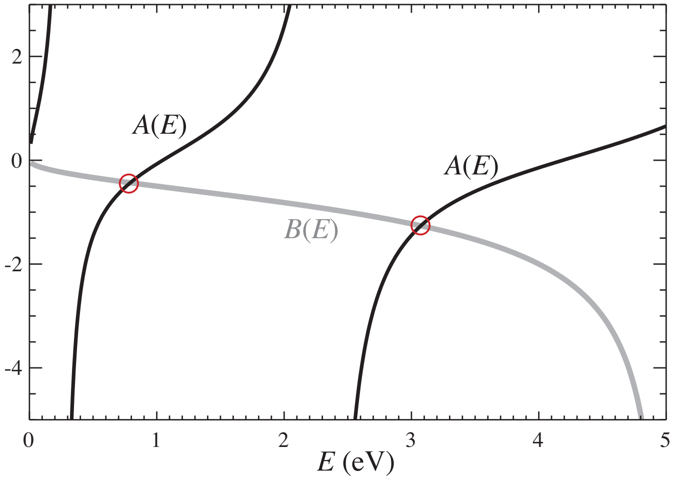
4. Look at Serway problem 6.23 again. Setting $L = 0.6$ nm, and $U = 5.0$ eV, find an approximate numerical result in eV for the energies of the first two quantum states, and find numerical values in nanometers for $1/k$ and $1/\alpha$ in the ground state.
   Answer: In solving problem 6.23, we found that by matching boundary conditions that solutions had to have the form

$$\tan\left(\frac{L\sqrt{2mE}}{\hbar}\right) = \tan\left(\frac{2\pi L\sqrt{2mc^2E}}{hc}\right) = -\sqrt{\frac{E}{U-E}}.$$

   Numerically,

$$\frac{2\pi L\sqrt{2mc^2}}{hc} = \frac{2\pi(0.6\text{ nm})\sqrt{2\cdot(511\times10^3\text{ eV})}}{1240\text{ eV}\cdot\text{nm}} = 3.07\text{ eV}^{-1/2} = 3.07/\sqrt{\text{eV}}$$

   so we want to plot $A(E) = \tan(\sqrt{E}\cdot 3.07)$ versus $B(E) = -\sqrt{E/(5.0-E)}$ over a range of values of E between 0 and $U = 5.0$ eV:



   As can be seen from the figure, there are two places where these two curves intersect: at $E \simeq 0.8$ eV, and $E \simeq 3.1$ eV. Those intersections give the energy of the ground state and the first excited state, respectively. We then have

$$\frac{1}{k} = \frac{\hbar}{\sqrt{2mE}} = \frac{hc}{2\pi\sqrt{2mc^2E}} = \frac{1240\text{ eV}\cdot\text{nm}}{2\pi\sqrt{2\cdot(511\times10^3\text{ eV})\cdot(0.8\text{ eV})}} = 0.22\text{ nm}$$

   and

$$\frac{1}{\alpha} = \frac{hc}{2\pi\sqrt{2mc^2(U-E)}} = \frac{1240\text{ eV}\cdot\text{nm}}{2\pi\sqrt{2\cdot(511\times10^3\text{ eV})\cdot(5.0-0.8\text{ nm})}} = 0.095\text{ nm}$$

5. Serway 6.28
   Answer: We want to use Serway Eq. 6.32 which says that the expected value $\langle f\rangle$ of a function $f(x)$ is found from

$$\langle f\rangle = \int_{-\infty}^{\infty} f(x)\,P(x)\,dx$$

where we have substituted $P(x)$ as a probability for $\psi^2(x)$. If $P(x) = 1/L$ we have

$$\begin{aligned} \langle x \rangle &= \int_0^L x \, \frac{1}{L} \, dx = \frac{1}{L}(\frac{x^2}{2})|_0^L) = \frac{L}{2} \\ \langle x^2 \rangle &= \int_0^L x^2 \, \frac{1}{L} \, dx = \frac{1}{L}(\frac{x^3}{3})|_0^L = \frac{L^2}{3} \end{aligned}$$

The quantum particle in a box example of problem 6.15 had $\langle x \rangle = L/2$ as we have here, but a different result of $\langle x^2 \rangle = L^2/3 - L^2/(2\pi^2)$ so it differs by a factor of $3/(2\pi^2) = 0.15$ or 15%.

6. 6.33

   Answer: **A).** In classical terms, we can think of the particle rolling forward the same amount of time that it rolls backwards. In wave propagation terms, a standing wave is an equal mix of forward and backward propagating waves. Both pictures tell us that the average momentum is zero, or $\langle p \rangle = 0$.

   B). The ground state energy of a quantum mechanical harmonic oscillator is $\frac{1}{2}\hbar\omega$. This energy is evenly distributed between kinetic energy $p^2/(2m)$ and potential energy $(1/2)kx^2 = (1/2)m\omega^2 x^2$. We can therefore say

   $$\frac{p^2}{2m} = (\frac{1}{2})\frac{1}{2}\hbar\omega \qquad \Rightarrow \qquad p^2 = \frac{m\hbar\omega}{2}$$

   from which we can infer $\langle p^2 \rangle = m\hbar\omega/2$. **C).** We can then find

   $$\Delta p = \sqrt{\langle p^2 \rangle - (\langle p \rangle)^2} = \sqrt{\frac{m\hbar\omega}{2} - (0)^2} = \sqrt{\frac{m\hbar\omega}{2}}.$$

   Let's think of of the units: $\hbar\omega$ is an energy with units of force times distance, or mass times acceleration times distance, or mks units of (kg) $\cdot$ $(\frac{m}{/}\mathrm{s}^2)$ $\cdot$ (m). Multiplying by mass and taking the square root gives kg·m/s which makes sense for momentum which is mass times velocity.

7. A 100 kg person is moving at a slow speed of 0.1 meters per second. What is the likelihood that they will tunnel through a 2 nanometer thick barrier, 2 meters tall? (Use mgh for the potential energy). If they try to pass through the wall once per second, will they likely tunnel through the wall over the 13 billion year lifetime of the universe?

   Answer: The wavefunction amplitude declines by $\exp[-x/\delta]$ so the probably declines by $(\exp[-x/\delta])^2 = \exp[-2x/\delta]$. The tunneling distance δ is

   $$\begin{aligned} \delta &= \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \frac{h}{\sqrt{8\pi^2 m(U_0 - E)}} = \frac{h}{\sqrt{8\pi^2 m(mgh - \frac{1}{2}mv^2)}} \\ &= \frac{h}{\sqrt{8\pi^2 m^2(gh - \frac{1}{2}v^2)}} = \frac{6.63 \times 10^{-34}}{\sqrt{8\pi^2 \cdot (100)^2 \cdot (9.80 \cdot 2 - \frac{1}{2}100 \cdot (0.1)^2}} = 1.7 \times 10^{-37} \text{ meters.} \end{aligned}$$

   We therefore have $2x/\delta = 2 \cdot (2 \times 10^{-9} \text{ m})/(1.7 \times 10^{-37} \text{ m}) = 2.4 \times 10^{28}$ so the probability $\exp[-2x/\delta]$ of being just on the other side of the wall an exceedingly small number. Now the lifetime of the universe is about

$$(13 \times 10^9 \text{ years})(365 \cdot 24 \cdot 60 \cdot 60 \, \frac{\text{sec}}{\text{year}}) = 4.1 \times 10^{17} \text{ seconds}.$$

and if the person were to hit the wall once per second they would have had $N = 4.1 \times 10^{17}$ tries to tunnel through the wall. The probability of their success is $P = N \exp[-2x/\delta]$ or

$$\log(P) = \log(N) - 2x/\delta = \log(4.1 \times 10^{17}) - 2.4 \times 10^{28} = 40.6 - 2.4 \times 10^{28}.$$

so you can see that even large multiples of the lifetime of the universe are completely overwhelmed by the low probability of success in tunneling for any try. Sure, it's possible in theory, but if it ain't gonna happen in a million times the lifetime of the universe, you can pretty much say it really ain't gonna happen. . .