PHY 251 Fall 2009: homework problem set 6, due in the PHY 251 drop box in room A-129 by noon on Friday, Oct. 23.

1. Serway 5.20
   Answer: If the wavelength is uncertain to $\Delta\lambda/\lambda = 10^{-6}$, that means that after 10^6 waves we don't know if we have half a wave or a full wave left over. That means the position is defined to $10^6\lambda$ or $10^6 \cdot (6000 \times 10^{-10}\text{ m})$ or $10^6 \cdot (6 \times 10^{-7}\text{ m})$ or 0.6 meters.

2. Serway 5.25
   Answer: From the energy-time version of the uncertainty principle, we can associate a mean lifetime τ as giving an energy width of

$$\Delta E \approx \frac{\hbar}{2\tau} = \frac{h}{4\pi\tau} = \frac{(6.63 \times 10^{-34}\text{ J}\cdot\text{s})(1\text{ eV}/1.6 \times 10^{-19}\text{ J})}{4\pi \cdot 10^{-10}\text{ s}} = 3.3 \times 10^{-6}\text{ eV}.$$

   This is too small an energy blur to measure with a detector with 5 eV energy resolution.

3. Serway 5.26.
   Answer: The full-width half-maximum distribution (FWHM) is about 5 bins across, or FWHM=$5 \cdot (25\text{ MeV}) = 125\text{ MeV}/c^2$ or in terms of energy $E = mc^2$ a FWHM value of 125 MeV. Now this looks like a Gaussian distribution of the form $\exp[-(x - x_0)^2/(2\sigma^2)]$. The half-width at half max point is x_{HW} such that the Gaussian has a value of 1/2, or (measuring distances x from the Gaussian center x_0) we have

$$\begin{aligned} \exp(-\frac{x_{\text{HW}}^2}{2\sigma^2}) = \frac{1}{2} \qquad \Rightarrow \qquad \exp(\frac{x_{\text{HW}}^2}{2\sigma^2}) &= 2 \\ \frac{x_{\text{HW}}^2}{2\sigma^2} = \ln 2 \qquad \Rightarrow \qquad x_{\text{HW}} &= \sqrt{2\,\ln 2}\,\sigma \\ \text{FWHM} = 2x_{\text{HW}} &= 2\sqrt{2\,\ln 2}\,\sigma = 2.35\,\sigma \end{aligned}$$

   so we'll characterize the 1σ uncertainty in energy as

$$\Delta E = \sigma = \frac{\text{FWHM}}{2.35} = \frac{125\text{ MeV}}{2.35} = 53\text{ MeV}.$$

   The lifetime is thus

$$\Delta t = \frac{\hbar}{2\Delta E} = \frac{6.58 \times 10^{-16}\text{ eV}\cdot\text{sec}}{2(53 \times 10^6\text{ eV})} = 6.2 \times 10^{-24}\text{ seconds}$$

   where this Δt represents a Gaussian σ value; the FWHM lifetime is 2.35σ or 14.57×10^{-24} seconds or 1.5×10^{-23} seconds when we use the sensible number of digits of precision based on our rough estimate of FWHM in mass.

4. Serway 5.32
   Answer: The frequency comes from $E = hf$, so

$$f = \frac{1.8\text{eV}}{4.1 \times 10{-}15\text{ eV}\cdot\text{s}} = 4.4 \times 10^{14}\text{Hz}.$$

The wavelength is $\lambda = hc/E = (1240/1.8) = 680$ nm which is red. The energy uncertainty is

$$\Delta E \approx \frac{h}{4\pi\tau} = \frac{4.1\times10^{-15}}{4\pi\cdot2.0\times10^{-6}} = 1.6\times10^{-10}\text{ eV}$$

5. Serway 6.8
   Answer: The velocity v is

$$v = (0.100\times10^{-9}\,\frac{\text{m}}{\text{yr}})\cdot\frac{1\text{ yr}}{365\text{ d}}\cdot\frac{1\text{ d}}{24\text{ h}}\cdot\frac{1\text{ h}}{3600\text{ s}} = 3.17\times10^{-18}\text{ m/s}.$$

   The bead has a kinetic energy of $E = (1/2)mv^2$ so we can find the quantum number n from

$$\begin{aligned} E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2} \\ n^2 &= \frac{2mL^2E}{(h/2)^2} = \frac{8mL^2(1/2)mv^2}{h^2} = \left(\frac{2mLv}{h}\right)^2 \\ n &= \frac{2\cdot(5.00\times10^{-3}\text{ kg})\cdot(0.200\text{ m})\cdot(3.17\times10^{-18}\text{ m/s})}{6.63\times10^{-34}} = 9.6\times10^{12} \end{aligned}$$

   so even this incredibly light and lethargic object is in a highly excited quantum state!

6. Serway 6.9
   Answer: The energy of the photon is given by

$$E_\gamma = \frac{\pi^2\hbar^2}{2mL^2}\cdot(2^2-1^2) = \frac{3\pi^2h^2}{8\pi^2mL^2} = \frac{3}{8}\,\frac{(hc)^2}{mc^2\,L^2} = \frac{3}{8}\,\frac{(1240\times10^{-9}\text{ eV}\cdot\text{m})^2}{(938.3\times10^6\text{ eV})\cdot(10^{-14}\text{ m})^2}$$

   or $E_\gamma = 6.1\times10^6$ eV or 6.1 MeV. This is known as a gamma ray (a photon with an energy of greater than about 100 keV).

7. Serway 6.10
   Answer: Again, the energy of states in an infinite quantum well is

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = n^2\frac{h^2}{8mL^2} = n^2\frac{(hc)^2}{8\,mc^2\,L^2} = n^2\frac{(1240\text{ eV}\cdot\text{nm})^2}{8\cdot(511\times10^3\text{ eV})\cdot(0.100\text{ nm})^2} = n^2(37.6\text{ eV}).$$

   The first four energy states are thus [37.6,150.4,338.4,601.6] eV, and the wavelengths involving all transitions that could take place from $n=4$ down to $n=1$ are

$$\begin{aligned} \lambda &= \frac{hc}{\Delta E} = \frac{1240\text{ eV}\cdot\text{nm}}{[4^2-3^2,4^2-2^2,4^2-1^2,3^2-2^2,3^2-1^2,2^2-1^2](37.6\text{ eV})} \\ &= [4.71,2.75,2.20,6.60,4.12,11.0]\text{ nm}. \end{aligned}$$

8. Serway 6.12
   Answer: The photon energy $E_\gamma = (1240/694.3) = 1.786$ eV must be given by

$$E_\gamma = \;=\; \frac{\pi^2\hbar^2}{2mL^2}(2^2-1^2)$$

$$
\begin{aligned}
L^2 &= \frac{3\pi^2 h^2}{8\pi^2 m E_\gamma} = \frac{3(hc)^2}{8mc^2\, E_\gamma} \\
L &= \frac{\sqrt{3}\,(hc)}{\sqrt{8mc^2\, E_\gamma}} = \frac{\sqrt{3}\,(1240\times 10^{-9}\ \text{eV}\cdot\ \text{m})}{\sqrt{8\cdot(511\times 10^3\ \text{eV})\cdot(1.786\ \text{eV})}} = 7.9\times 10^{-10}\ \text{m}
\end{aligned}
$$

or 0.79 nm.

9. Serway 6.17. For this problem, recall that $\int |\psi|^2 = 1$ over all space; the integral of $|\psi|^2$ over a subset of allowed positions tells you the probability of being within that subset of allowed positions.
Answer: (a) The wavefunctions and probabilities are shown in Serway Fig. 6.9.
(b) The wavefunction of the $n = 1$ state is $\psi = \sqrt{2/L}\sin(\pi x/L)$. We can check the normalization N by integrating from $x = 0$ to $x = L$:

$$N = \int_{x=0}^{x=L} \Big|\sqrt{\frac{2}{L}}\sin(\frac{\pi x}{L})\Big|^2\, dx = \frac{2}{L}\int_{x=0}^{x=L} \sin^2(\frac{\pi x}{L})\, dx$$

Now let $y \equiv \pi x/L$ so $dy = (\pi/L)\,dx$. This gives $dx = (L/\pi)dy$, and the integration limit $x = 0$ becomes $y = 0$ while the limit $x = L$ becomes $y = \pi$:

$$
\begin{aligned}
N &= \frac{2}{L}\frac{L}{\pi}\int_{y=0}^{y=\pi} \sin^2(y)\, dy = \frac{2}{\pi}\{\frac{y}{2} - \frac{1}{4}\sin(2y)\}\Big|_{y=0}^{y=\pi} \\
&= \frac{2}{\pi}\Big[\{\frac{\pi}{2} - \frac{1}{4}\sin(2\pi)\} - \{\frac{0}{2} - \frac{1}{4}\sin(0)\}\Big] = \frac{2}{\pi}\Big[\{\frac{\pi}{2} - 0\} - \{0 - 0\}\Big] = 1
\end{aligned}
$$

and since $N = 1$ the wavefunction is indeed properly normalized. To calculate the probability P of being within the range $x = 0.15$ nm to $x = 0.35$ nm (or $x = 0.15L$ to $x = 0.35L$, or $y = 0.15\pi$ to $y = 0.35\pi$) we just modify the definite integral result:

$$
\begin{aligned}
P &= \frac{2}{\pi}\{\frac{y}{2} - \frac{1}{4}\sin(2y)\}\Big|_{y=0.15\pi}^{y=0.35\pi} = \frac{2}{\pi}\Big[\{\frac{0.35\pi}{2} - \frac{1}{4}\sin(2\cdot 0.35\pi)\} - \{\frac{0.15\pi}{2} - \frac{1}{4}\sin(2\cdot 0.15\pi)\}\Big] \\
&= \frac{2}{\pi}\Big[0.1\pi - \frac{1}{4}\{\sin(0.7\pi) - \sin(0.3\pi)\}\Big] = 0.20
\end{aligned}
$$

so the particle has a 20% chance of being in this region.
(c) For the $n = 2$ state, we have $\psi = A\sin(2\pi x/L)$ and

$$1 = \int_{x=0}^{x=L} \Big|A\sin(\frac{2\pi x}{L})\Big|^2\, dx = A^2\int_{x=0}^{x=L} \sin^2(\frac{2\pi x}{L})\, dx.$$

Now let $y \equiv 2\pi x/L$ which gives $dy = (2\pi/L)\,dx$ and $dx = (L/2\pi)dy$. The integration limit $x = 0$ becomes $y = 0$, while the integration limit $x = L$ becomes $y = 2\pi$:

$$
\begin{aligned}
1 &= A^2\frac{L}{2\pi}\int_{y=0}^{y=\pi} \sin^2(y)\, dy = A^2\frac{L}{2\pi}\{\frac{y}{2} - \frac{1}{4}\sin(2y)\}\Big|_{y=0}^{y=2\pi} \\
&= A^2\frac{L}{2\pi}\Big[\{\frac{2\pi}{2} - \frac{1}{4}\sin(4\pi)\} - \{\frac{0}{2} - \frac{1}{4}\sin(0)\}\Big] = A^2\frac{L}{2\pi}\left[\{\pi - 0\} - \{0 - 0\}\right] = A^2\frac{L}{2}
\end{aligned}
$$

so we find out that $A = \sqrt{2/L}$ just like the case with $n = 1$. For the range from $x = 0.15L$ to $0.35L$, we have $y = 0.15 \cdot 2\pi$ to $0.35 \cdot 2\pi$, and the probability P becomes

$$\begin{aligned} P &= \frac{2}{L}\frac{L}{2\pi}\left[\{\frac{0.35 \cdot 2\pi}{2} - \frac{1}{4}\sin(2 \cdot 0.35 \cdot 2\pi)\} - \{\frac{0.15 \cdot 2\pi}{2} - \frac{1}{4}\sin(2 \cdot 0.15 \cdot 2\pi)\}\right] \\ &= \frac{1}{\pi}\left[0.2\pi - \frac{1}{4}\{\sin(0.35 \cdot 4\pi) - \sin(0.15 \cdot 4\pi)\}\right] = 0.351 \end{aligned}$$

so the particle spends a little bit more time in this particular region in the $n = 2$ case.

(d) The state energies are

$$E = n^2 \frac{\pi^2 \hbar^2}{2mL^2} = n^2 \frac{(hc)^2}{8mc^2L^2} = n^2 \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8 \cdot (511 \times 10^3 \text{ eV}) \cdot (1.0 \text{ nm})^2} = n^2\,(0.376 \text{ eV})$$

so the state energies are $\{0.376, 1.504\}$ eV for $n = \{1, 2\}$.

10. Serway 6.23

Answer: Because the potential goes to ∞ at $x = 0$, we need to have $\psi(x = 0) = 0$ so that we don't have a divergence at $U(x \leq 0) \cdot \psi(x \leq 0)$. In region I ($0 \leq x \leq L$) we have $U = 0$ so a valid wavefunction is (see p. 201)

$$\psi_I = A\sin(kx) \qquad \text{with} \qquad k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{\sqrt{2mE}}{\hbar}.$$

In region II ($x > L$) we are in a classically disallowed region so we will have an exponentially decaying wave function of the form (see Serway Eq. 6.21)

$$\psi_{II} = Be^{-\alpha(x-L)} \qquad \text{with} \qquad \alpha = \frac{\sqrt{2m(U-E)}}{\hbar}.$$

Our boundary conditions are

$$\psi_I(x = L) = \psi_{II}(x = L) \qquad \text{and} \qquad \frac{d\psi_I}{dx}|_{x=L} = \frac{d\psi_{II}}{dx}|_{x=L}$$

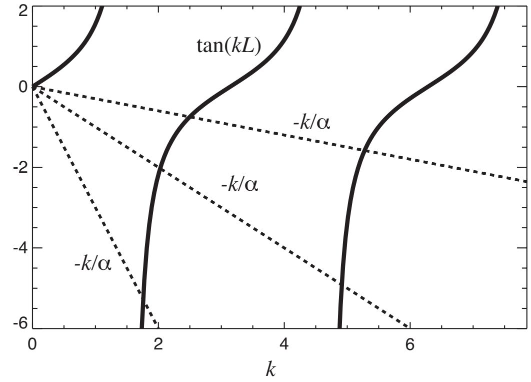
or

$$A\sin(kL) = Be^{-\alpha \cdot 0} = B \qquad \text{and} \qquad Ak\cos(kL) = -\alpha Be^{-\alpha \cdot 0} = -\alpha B$$

and if we divide these two equations we have

$$\frac{A\sin(kl)}{Ak\cos(kl)} = \frac{B}{-\alpha B} \qquad \Rightarrow \qquad \tan(kL) = -\frac{k}{\alpha}.$$

This does not give us a simple solution, but we can learn something from a graphical solution of how $\tan(kL)$ scales with k, versus how $-k/\alpha$ scales with k:



As you can see, no matter what slope $-1/\alpha$ we pick, we're going to have only discrete points where $\tan(kL)$ intersects with $-k/\alpha$. Once we identify a point k where such a solution occurs, we can find E from $k = \sqrt{2mE}/\hbar$ or better yet we can write $\tan(kL) = -k/\alpha$ as

$$\tan\left(\frac{L\sqrt{2mE}}{\hbar}\right) = -\sqrt{\frac{E}{U-E}}.$$

11. Serway 6.32
Answer: The lowest energy solution wavefunction is $\psi = C_0 e^{-\alpha x^2}$ so we have

$$\langle x\rangle = \int_{-\infty}^{\infty} x\,(C_0 e^{-\alpha x^2})^2\,dx = C_0^2 \int_{-\infty}^{\infty} x\,e^{-2\alpha x^2}\,dx.$$

If we set $y \equiv e^{-2\alpha x^2}$ we have $dy = -4\alpha x\,e^{-2\alpha x^2}\,dx$ so the above integral becomes

$$\langle x\rangle = -\frac{C_0^2}{4\alpha}\int_{x=-\infty}^{x=\infty} dy = -\frac{C_0^2}{4\alpha} e^{-2\alpha x^2}|_{-\infty}^{\infty} = 0$$

because $e^{-2\alpha\infty} \rightarrow 0$ no matter what the value of α is. For $\langle x^2\rangle$ we have

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\,(C_0 e^{-\alpha x^2})^2\,dx = C_0^2 \int_{-\infty}^{\infty} x^2\,e^{-2\alpha x^2}\,dx$$

and if we make the substitution $a \equiv 2\alpha$ we have

$$\langle x^2\rangle = C_0^2 \int_{-\infty}^{\infty} x^2\,e^{-ax^2}\,dx = C_0^2\,\frac{1}{4a}\sqrt{\frac{\pi}{a}} = C_0^2\,\frac{1}{8\alpha}\sqrt{\frac{\pi}{2\alpha}}.$$

Now let's use the result $\alpha = m\omega/(2\hbar)$ from Serway Eq. 6.27, and $C_0^2 = \sqrt{(m\omega)/(\pi\hbar)}$ from Serway Example 6.10 to find

$$\langle x^2\rangle = C_0^2 \frac{1}{8\alpha}\sqrt{\frac{\pi}{2\alpha}} = \frac{\sqrt{m\omega}}{\sqrt{\pi\hbar}}\,\frac{2\hbar}{8m\omega}\sqrt{\frac{2\pi\hbar}{2m\omega}} = \frac{\hbar}{4m\omega}.$$

Let's look at the units of this. The units of $\hbar$ are of energy times time, or kg·m^2/s. The units of $m\omega$ is in kg/s. Therefore the units of $\hbar/(4m\omega)$ are meters squared as expected for $\langle x^2\rangle$. Finally, we can use Serway Eq. 6.34 to find

$$\Delta x = \sqrt{\langle x^2\rangle - (\langle x\rangle)^2} = \sqrt{\frac{\hbar}{4m\omega} - (0)^2} = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}.$$