PHY 251 Fall 2009: homework problem set 4, due in the PHY 251 drop box in room A-129 by noon on Friday, Oct. 9.

1. Do Serway problem 4.2 on p. 146. This problem illustrates how electrochemistry experiments already gave some idea about the discrete charge of the electron, once you believe in discrete atoms and know Avogadro's number!
   Answer: Copper sulfate involves Cu^{+2} and SO_4^{-2}, so there are two valence electrons per atom. We then find that the fraction of a mole is

   $$\frac{(1\ \text{C/sec})\cdot(3600\ \text{sec})}{(96,500\ \text{C/mol})\cdot 2} = 0.0187\ \text{mol}.$$

   The number of coppper atoms deposited is then

   $$(0.0187\ \text{mol})\cdot(6.02\times 10^{23}\ \text{atoms/mol}) = 1.13\times 10^{22},$$

   the weight of a copper atom is

   $$\frac{1.185\times 10^{-3}\ \text{kg}}{1.13\times 10^{22}\ \text{atoms}} = 1.05\times 10^{-25}\ \text{kg},$$

   and the molar mass of copper is

   $$\frac{1.185\ \text{g}}{0.0187\ \text{mol}} = 63.4\ \text{g/mol}$$

   which is about right (of course they gave us numbers that would work out!).

2. Do Serway problem 4.3.
   Answer: In the apparatus, we have

   $$F_y = q\frac{V}{d} = m\frac{\Delta v_y}{\Delta t}$$

   and we also have $v_x = \ell/\Delta t$ or $\Delta t = \ell/v_x$. Therefore we can write

   $$\Delta v_y = \frac{qV\,\Delta t}{md} = \frac{qV}{md}\frac{\ell}{v_x}.$$

   We can then figure out the angle θ:

   $$\tan\theta = \frac{v_y}{v_x} = \frac{qV\ell}{mdv_x^2} \qquad \text{so} \qquad \frac{q}{m} = \frac{dv_x^2\tan\theta}{V\ell}.$$

   Now from the magnetic field cancelling the electric field we can say

   $$\frac{qV}{d} = qv_xB \qquad \text{so} \qquad v_x = \frac{V}{Bd} = \frac{2000}{4.57\times 10^{-2}\cdot 0.02} = 2.19\times 10^6\ \text{m/s}$$

   which means relativistic effects are small ($\gamma = 1.000\,027$). We can also substitute this result for velocity into our expression for q/m:

   $$\frac{q}{m} = \frac{d}{V\ell}\frac{V^2}{B^2d^2}\tan\theta = \frac{V}{ldB^2}\tan\theta = \frac{2000}{0.10\cdot 0.02\cdot(4.57\times 10^{-2})^2}\tan(0.2) = 9.7\times 10^7\ \text{C/kg}.$$

   A proton has $q/m = 1.6\times 10^{-19}/1.67\times 10^{-27} = 9.7\times 10^7$ so it seems like we have a good candidate. . .

3. Do Serway problem 4.4.
Answer: The electric field the electron experiences when it travels between the two plates is $E_y = V/d$, and the acceleration it experiences is $a = F/m = qE/m = qV/(md)$. Since we expect $v_y \ll v_x$, we can say that the time that the electron experiences this acceleration is $\Delta t = \ell/v_x$. Consequently the electron receives a velocity kick in the $\hat{y}$ direction of

$$v_y = a_y \cdot \Delta t = \frac{qV}{md} \cdot \frac{\ell}{v_x}$$

so that its angle θ leaving the field region is $\theta = v_y/v_x = y_2/D$ (the latter by geometry in the small angle limit) or

$$\theta = \frac{y_2}{D} = \frac{v_y}{v_x} = \frac{qV}{md}\frac{\ell}{v_x^2}$$

which can be solved for y_2 to give

$$y_2 = \frac{qVD\ell}{mdv_x^2}.$$

We can calculate y_1 from the distance traveled under constant acceleration or

$$y_1 = \frac{1}{2}a(\Delta t)^2 = \frac{1}{2}\frac{qV}{md}(\frac{\ell}{v_x^2})^2 = \frac{qV\ell^2}{2mdv_x^2}.$$

Now since $y = y_1 + y_2$ we have

$$\begin{aligned} y &= y_1 + y_2 = \frac{qV\ell^2}{2mdv_x^2} + \frac{qV\,D\ell}{mdv_x^2} = \frac{q}{m}\frac{V\ell}{dv_x^2}(D + \ell/2) \\ \frac{q}{m} &= \frac{ydv_x^2}{V\ell(D + \ell/2)}. \end{aligned}$$

4. Rutherford scattering involves α particles being electrostatically repelled by the nucleus, so that the scattering obeys the $(\sin\phi/2)^{-4}$ trend. However, when particles get closer than a certain distance from the nucleus, they can be affected by nuclear forces instead. It is found that when α particles scatter off of a tungsten foil that they depart from the Rutherford scattering angle at energies above 42 MeV. Use this information to estimate the size of the tungsten nucleus.
Answer: The electrostatic force is $F = kq_1q_2/r^2$ so the work needed to bring the charge of an α particle from infinitely far away to a distance r from the nucleus is

$$\begin{aligned} W &= \int_r^\infty F\,dx = kq_1q_2\int_r^\infty \frac{1}{r'^2}\,dr' = kq_1q_2\left(\frac{-1}{r'}\right)|_r^\infty \\ &= kq_1q_2\left[\frac{-1}{\infty} - \frac{-1}{r}\right] = \frac{q_1q_2}{4\pi\epsilon_0}\frac{1}{r}. \end{aligned}$$

When this equals the kinetic energy, the particle will stop and turn around or be deflected to the side, and Rutherford's expression (which is based solely on Coulomb repulsion) will hold. If instead the particle experiences some other force such as a nuclear binding force at this distance, then there will be deviations from Rutherford's law. So, what we want to do is to set the energy calculated above to be equal to the kinetic energy E_k of the α particle and solve for the distance r:

$$\begin{aligned} E_k &= \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r} \\ r &= \frac{q_1 q_2}{4\pi\epsilon_0 E_k} = \frac{(2e)(Ze)}{4\pi\epsilon_0 E_k} \\ &= \frac{2\cdot 74\cdot(1.602\times10^{-19})^2}{4\pi\cdot 8.85\times10^{-12}\cdot(42\times10^6\ \text{eV})\cdot(1.602\times10^{-19}\ \text{J/eV})} = 5.1\times10^{-15}\ \text{meters} \end{aligned}$$

where we have been careful to use mks units throughout to get an answer in mks units.

5. Construct an energy level diagram for the ground state and the first two excited states of the Li^{2+} ion (lithium with two electrons removed). Calculate the photon wavelengths associated with all possible transitions between these states (for those of you who know about selection rules, ignore them for this problem).
Answer: A Li^{2+} ion might start out life as a Li atom with $Z = 3$ protons in its nucleus and 3 electrons; take one electron away and you have singly ionized lithium or Li^{+}, and take two away and you have doubly ionized lithium or Li^{2+}. You are then back to a pretty accurate picture from the Bohr model, because you have just one electron to worry about. The energies are given by

$$E_n = -\frac{Z^2}{n^2}E_0 = -\frac{3^2}{n^2}\cdot 13.60\ \text{eV}$$

or $E_1 = -122.4$ eV for the ground state, $E_2 = -30.6$ eV for the first excited state, and $E_3 = -13.6$ eV for the second excited state. The transitions are 122.4-30.6=91.8 eV corresponding to $\lambda = 13.50$ nm, 122.4-13.6=108.8 eV corresponding to $\lambda = 11.40$ nm, and 30.6-13.6=17.0 eV corresponding to $\lambda = 72.9$ nm.

6. Serway 4.23
Answer: The radius of the electron's orbit is given by (Serway Eq. 4.35)

$$r_n = \frac{n^2}{Z}\frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = \frac{n^2}{Z}a_0 \qquad \text{with} \qquad a_0 \equiv \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 0.053\ \text{nm}.$$

For hydrogen's ground state we have $Z = 1$ and $n = 1$, so $r = 0.053$ nm. The velocity is found from Serway Eq. 4.24, which gives

$$\begin{aligned} l = rmv &= n\hbar \\ v &= \frac{n\hbar}{m_e}\frac{1}{r} = \frac{n\hbar}{m_e}\frac{Zm_e e^2}{n^2 4\pi\epsilon_0\hbar^2} = \frac{Z}{n}\frac{e^2}{2h\epsilon_0 c}c = \frac{Z}{n}\alpha c \end{aligned}$$

or since $n = 1$ and $\alpha \simeq 1/137$, we have $v = \alpha c \simeq c/137$. This is non-relativistic, so the linear momentum is

$$p = mv = m\alpha c = \alpha\frac{mc^2}{c} = \frac{511\times10^3\ \text{eV/c}}{137} = 3.73\ \text{eV/c}.$$

The angular momentum is found from

$$l = n\hbar = 1 \cdot (6.582 \times 10^{-16}\ \text{eV} \cdot \text{sec})$$

The total energy is $E_n = -E_0 Z^2/n^2$ or just $-E_0 = -13.6$ eV. What you see in Serway Eq. 4.25 is that the total energy is equal to the sum of the kinetic plus potential, and from Eq. 4.27 you see that the total energy is equal to half of the potential. Therefore $E = K + U = U/2$ so $U = 2E = 2(-13.6\ \text{eV}) = -27.2$ eV, and $K = E - U = E - 2E = -E = -(-13.6\ \text{eV}) = +13.6$ eV.

7. Serway 4.28
Answer: Chromium has $Z = 24$. We first have to ask how much energy is made available by an electron dropping from the $n = 2$ state to the $n = 1$ state:

$$\Delta E = Z^2 E_0 \Big(\frac{1}{n_f^2} - \frac{1}{n_i^2}\Big) = (24)^2 \cdot 13.60 \cdot \Big(\frac{1}{1^2} - \frac{1}{2^2}\Big) = 5875\ \text{eV}.$$

(Note that we must have had a violent event, like absorption of an x-ray with an energy of at least $(24)^2 \cdot 13.60/1^2 = 7833$ eV, take place to rip out an electron from the $n = 1$ state to begin with. Note also that we are ignoring the fact that there is some screening of the nuclear charge by the *other* $n = 1$ electron; remember that we talked about $(Z-1)^2$ in Moseley's law? But let us plow ahead. We now have some energy we can spend; part of what we pay has to go to unbinding a $n = 4$ electron, and we can then spend what's left on kinetic energy. The binding energy of a $n = 4$ electron is

$$E_n = -\frac{Z^2}{n^2} E_0 = -\frac{24^2}{4^2} 13.6 = -490\ \text{eV}$$

(of course we're lying a bit here because there is some screening of the nuclear charge by the other electrons, but this is at least a ballpark estimate). So, we have 5875 eV to give, 490 eV of it has to go to removing the $n = 4$ electron, and 5875-490=5386 eV is left over to go into kinetic energy of the ejected electron.

8. Serway 4.43
Answer: The atom mass is $m = 10^{-25}$ kg, and it has an initial velocity of v_1. The photon has an energy of $E = hc/\lambda = 1240/500 = 2.48$ eV. If we say that the atom is moving in the $+x$ direction and the photon comes in head-on in the $-x$ direction, conservation of momentum tells us about the momentum of the atom after the collision:

$$\begin{aligned} mv_1 - \frac{E}{c} &= mv_2 \\ mv_2 - mv_1 &= -\frac{E}{c} \\ \Delta v = v_2 - v_1 &= -\frac{E}{mc} = \end{aligned}$$

If it takes 10^{-8} seconds before the atom deexcites by spontaneous emission, then the atom is next able to absorb a $\lambda = 500$ nm photon at that same time later. As a result, the acceleration is

$$a = -\frac{\Delta v}{\Delta t} = \frac{E/mc}{\Delta t} = -\frac{E}{mc\,\Delta t} = -\frac{(2.48\ \text{eV}) \cdot (1.6 \times 10^{-19}\ \text{J/eV})}{(10^{-25}\ \text{kg}) \cdot (2.99 \times 10^{8}\ \text{m/s}) \cdot (10^{-8}\ \text{s})} = -1.3{\times}10^{6}\ \text{m/s}^2$$

Because the spontaneously emitted photons have no preferred direction, they produce no preferred momentum kick; that is, they average out to zero acceleration. Now that we know the acceleration, we can figure out that the atoms slow down over a distance of

$$\begin{aligned} v_f^2 &= v_0^2 + 2a(x - x_0) \\ (x - x_0) &= \frac{v_f^2 - v_0^2}{2a} = \frac{0^2 - (10^3 \text{ m/s})^2}{2(-1.3 \times 10^6)} = 0.38 \text{ m} \end{aligned}$$

9. Serway 4.44

Answer: In the Bohr model, the wavelengths corresponding to state transitions are given by

$$\lambda = \frac{hc}{\Delta E} = \frac{hc}{Z^2 E_0} \frac{1}{1/n_f^2 - 1/n_i^2} = a \frac{hc}{Z^2 E_0}$$

where $n_i > n_f$. Regarding the factor $a = 1/(1/n_f^2 - 1/n_i^2)$, the initial state can range from $n_i \to \infty$ to $n_i = n_f + 1$. Therefore the ratio of wavelengths for the series with the same final state is

$$\begin{aligned} \frac{\lambda_{\text{shortest}}}{\lambda_{\text{longest}}} &= \frac{1/n_f^2 - 1/(n_f + 1)^2}{1/n_f^2 - 1/\infty} = \frac{1 - n_f^2/(n_f^2 + 2n_f - 1)}{1 - 0} \\ &= 1 - n_f^2/(n_f^2 + 2n_f + 1) = 1 - \frac{1}{1 + 2/n_f + 1/n_f^2} \end{aligned}$$

which has values of 0.75 for $n_f = 1$, 0.55 for $n_f = 2$, 0.44 for $n_f = 3$, 0.36 for $n_f = 4$, 0.31 for $n_f = 5$, and so on. Since in our case we have a wavelength ratio of $22.8/63.3 = 0.36$, we see that we have $n_f = 4$. We can then find Z using $\lambda = 22.8$ nm for the $n_i \to \infty$ and $n_f = 4$ transition as

$$\lambda = \frac{hc}{\Delta E} = \frac{hc}{Z^2 E_0 / n_f^2} \qquad \to \qquad Z = n_f \sqrt{\frac{hc}{\lambda E_0}} = 4 \sqrt{\frac{1240 \text{ eV} \cdot \text{nm}}{(22.8 \text{ nm}) \cdot (13.6 \text{ eV})}} = 8.00$$

so it's an Oxygen atom we're talking about.