PHY 251 Fall 2009: homework problem set 3, due PHY 251 drop box in room A-129 by noon on Friday, Sep. 25.

1. A proton with a kinetic energy of 10 GeV collides with an antiproton at rest. They annihilate each other, and produce two identical-energy gamma rays (very energetic photons) that come out at angles θ above and below, respectively, the direction of the proton's motion. Calculate the energy E and momentum p of the gamma rays, and the angle θ.
   Answer: Conservation of energy gives
$$(E_K + m_p c^2) + m_p c^2 = 2E_\gamma$$
   because we go from a proton with kinetic energy $E_K = 10$ GeV and the mass-energy of the proton plus the antiproton (same mass as proton), to two photons of identical energy E_γ. Since $m_p = 938.3$ MeV/c^2 we have $E_\gamma = (10 + 0.9383 + 0.9383)/2 = 5.94$ GeV. Note for the proton we have
$$\begin{aligned} E_K &= (\gamma - 1) m_p c^2 \\ \gamma &= 1 + \frac{E_k}{m_p c^2} = 1 + \frac{10}{0.9383} = 11.66 \end{aligned}$$
   from which we can find $\beta = \sqrt{1 - 1/\gamma^2} = 0.996$. At the same time, conservation of momentum gives
$$\begin{aligned} p_p &= 2p_\gamma \cos\theta \\ \gamma m_p v &= 2\frac{E_\gamma}{c}\cos\theta \\ \gamma\beta m_p c^2 &= 2E_\gamma \cos\theta \\ \cos\theta &= \frac{\gamma\beta m_p c^2}{2E_\gamma} \\ \theta &= \cos^{-1}\Big(\frac{\gamma\beta m_p c^2}{2E_\gamma}\Big) = \cos^{-1}\Big(\frac{11.66 \cdot 0.996 \cdot 0.9383}{2 \cdot 5.94}\Big) = \cos^{-1}(0.703) = 23.5^\circ \end{aligned}$$

2. A atom of the radioactive isotope of carbon ^{14}C (mass $14.003\,242$ amu, where 1 amu=931.494 MeV/c^2) decays into an electron (mass $0.000\,549$ amu) and a singly ionized ^{14}N atom (mass $14.003\,074 - 0.000\,549$ amu). If the ^{14}C atom were at rest, and no other particles were involved, calculate the total energy released by this decay, and the energy and momentum of the electron and ^{14}N atoms after the decay.
   Answer: Let's have $m_C = 14.003\,242$ amu represent the mass of the ^{14}C atom before the decay, $m_N = 14.003\,074 - 0.000\,549$ am represent the mass of the ^{14}N atom minus an electron after the decay, and $m_e = 0.000\,549$ amu represent the mass of the electron. The total energy released in the decay is given by
$$\begin{aligned} \frac{\Delta E}{c^2} &= \Delta m = [m_C - m_N - m_e] = [14.003\,242 - (14.003\,074 - 0.000\,549) - 0.000\,549] \text{ amu} \\ &= [14.00\,3242 - 14.003\,074] \text{ amu} = 0.000\,168 \text{ amu} \\ \text{or } \Delta E &= (0.000\,168 \text{ amu}) \cdot (\frac{931.494 \text{ MeV}/c^2}{\text{amu}}) \cdot c^2 = 0.156 \text{ MeV} \end{aligned}$$

or 156 keV. Now even if all of that energy went to the electron it would have a value of $\gamma = 1 + E_k/mc^2$ that would be no more than about 1.3. OK, so start with a momentum of 0 and a total energy of $E = m_C c^2$. After the decay, conservation of momentum tells us that

$$(\gamma_N \beta_N) m_N c = (\gamma_e \beta_e) m_e c$$

while conservation of energy tells us

$$m_C c^2 = \gamma_N m_N c^2 + \gamma_e m_e c^2$$

so we have two equations with two unknowns of β_N and β_e (because of course γ is a function of β). Solving the second equation for γ_N gives

$$\gamma_N = \frac{m_C - \gamma_e m_e}{m_N}.$$

Now because we know that γ_e is no more than about 1.3, we can see that $\gamma_e m_e \ll m_C$ so that γ_N must be very close to 1. (And this makes sense classically, because if we put a spring between a truck and a tennis ball we know the tennis ball will fly away and the truck will not get much of a velocity kick at all!). Setting γ_N to 1, we find

$$\gamma_e = \frac{(m_C - m_N}{m_e} = \frac{14.003\,242 - (14.003\,074 - 0.000\,549}{0.000\,549} = 1.306$$

which gives $\beta_e = \sqrt{1 - 1/\gamma_e^2} = 0.643$ and we can then use the conservation of momentum equation with $\gamma_N = 1$ to find

$$\beta_N = \frac{\gamma_e \beta_e m_e}{m_N} = \frac{1.306 \cdot 0.643 \cdot 0.000\,549}{14.003\,074 - 0.000\,549} = 3.29 \times 10^{-5}$$

which is consistent with our assumption that $\gamma_N \simeq 1$! For the electron, we have

$$p_e = \gamma_e \beta_e m_e c = 1.306 \cdot 0.643 \cdot (511 \text{ keV}/c^2)c = 429 \text{ keV}/c$$

and $E_k = (\gamma - 1) m_e c^2 = (1.306 - 1)(511 \text{ keV}/c^2)c^2 = 156$ keV, while for the singly ionized ^{14}N atom we have

$$p_N = \gamma_N \beta_N m_N c = 1 \cdot (3.29 \times 10^{-5})(14.003\,074 - 0.000\,549 \text{ amu})(\frac{931.494 \text{ MeV}/c^2}{\text{amu}})c = 0.429 \text{ MeV}/c$$

or 429 keV/c (as expected) and

$$\begin{aligned} E_k &= (\gamma - 1) m_N c^2 \simeq \frac{1}{2} \beta_N^2 m_N c^2 \\ &= \frac{1}{2}(3.29 \times 10^{-5})^2 (14.003\,074 - 0.000\,549 \text{ amu})(\frac{931.494 \text{ MeV}/c^2}{\text{amu}})c^2 = 7.06 \times 10^{-6} \text{ MeV} \end{aligned}$$

or 7.06 eV, which is of course a very small fraction of the kinetic energy carried by the electron.

3. Do Serway problem 3.6.
Answer: The Planck length is

$$\Big(\frac{hG}{c^3}\Big)^{\frac{1}{2}} = \Big(\frac{(6.626 \times 10^{-34} \text{ kg} \cdot \text{m}^2/\text{s}) \cdot (6.673 \times 10^{-11} \text{m}^3/\text{kg}^2/\text{s}^2)}{(2.998 \times 10^8 \text{ m/s})^3}\Big)^{1/2} = 4.05 \times 10^{-35} \text{ m}$$

The Planck time works out to 1.35×10^{-43} seconds, and the Planck mass works out to 5.46×10^{-8} kg. These Planck scales turn out to provide natural scalings for theories of quantum gravity (quantum mechanics plus general relativity).

4. Look up the overall luminous efficiency of a 100 W incandenscent light bulb on Wikipedia. Assume that the power emitted as visible light is all emitted at $\lambda = 500$ nm. How many photons per second does this light bulb emit?
Answer: See `http://en.wikipedia.org/wiki/Incandescent_light_bulb` which indicates that a 100 W incandescent blub has an overall luminous efficiency of only 2.6%, so the light power is only 2.6 W. The photon energy is $E = hc/\lambda = (1240 \text{ eV} \cdot \text{nm})/(500 \text{ nm}) = 2.48$ eV. We then have a photon flux of

$$(2.6 \, \frac{\text{Joules}}{\text{sec}}) \cdot (\frac{1 \text{ photon}}{2.48 \text{ eV}}) \cdot (\frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ Joules}}) = 6.54 \times 10^{18} \, \frac{\text{photons}}{\text{sec}}.$$

5. The photocurrent produced from a metal target by $\lambda = 220$ nm light is driven to zero by a retarding potential of 3.05 Volts. What's the work function of the metal?
Answer: The stopping potential times an electron's charge is equal to the kinetic energy of ejected electrons with no stopping potential. Therefore we have

$$\varphi = \frac{hc}{\lambda} - E_k = \frac{1240 \text{ eV} \cdot \text{nm}}{220 \text{ nm}} - 3.05 \text{ eV} = 2.59 \text{ eV}$$

6. Photons of wavelength 440 nm are incident on a metal. Ejected electrons are then directed into a region with a 1.5×10^{-5} Tesla magnetic field, where their trajectory is bent into a circular arc with a radius of 15 cm. Find the work function of the metal.
Answer: The photoelectric effect produces very low energy electrons, so we can ignore relativity in this problem. When the electron follows a circular trajectory, we know that the magnetic field is at right angles to the electron's motion, or $qvB = mv^2/r$ which gives $v = qBr/m$ and therefore

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\,\frac{q^2B^2r^2}{m^2} = \frac{q^2B^2r^2}{2m} = \frac{(1.602 \times 10^{-19})^2(1.5 \times 10^{-5})^2(0.15)^2}{2(9.11 \times 10^{-31})} = 7.1 \times 10^{-20} \text{ Joules}$$

or 0.44 eV (which is very non-relativistic!). The photon energy was $E = hc/\lambda = (1240/440) = 2.82$ eV, so the work function is $\varphi = h\nu - E_k = 2.38$ eV.

7. Some 2.00 MeV gamma rays are incident on a target. At what angle will 0.50 MeV Compton scattered photons be observed?
Answer: We can rearrange the Compton formula to find the scattering angle as follows:

$$\begin{aligned} \lambda_s - \lambda_0 &= \frac{h}{m_e c}(1 - \cos\theta) \\ m_e c^2(\frac{\lambda_s}{hc} - \frac{\lambda_0}{hc}) = m_e c^2(\frac{1}{E_s} - \frac{1}{E_0}) &= 1 - \cos\theta \end{aligned}$$

giving

$$\begin{aligned}\theta &= \arccos\Big[1 + m_e c^2 (\frac{1}{E_0} - \frac{1}{E_s})\Big] \\ &= \arccos\Big[1 + (0.511 \text{ MeV})(\frac{1}{2.00 \text{ MeV}} - \frac{1}{0.5 \text{ MeV}})\Big] = \arccos[0.2335] = 76.5^\circ\end{aligned}$$

8. A 10 keV photon is Compton scattered through an angle of 20°. What's the energy of the scattered photon? (Think of how to calculate it accurately).
Answer: Let's say that $E_s = E_0 - \Delta E$, so that

$$\lambda_s = \frac{hc}{E_s} = \frac{hc}{E_0 - \Delta E} = \frac{hc}{E_0}(1 - \frac{\Delta E}{E_0})^{-1} \simeq \lambda_0(1 + \frac{\Delta E}{E_0}).$$

We can now substitute this into the Compton formula:

$$\begin{aligned}\lambda_s - \lambda_0 &= \frac{h}{m_e c}(1 - \cos\theta) \\ \lambda_0(1 + \frac{\Delta E}{E_0}) - \lambda_0 &= \frac{hc}{m_e c^2}(1 - \cos\theta) \\ \frac{hc}{E_0}\frac{\Delta E}{E_0} &= \frac{hc}{m_e c^2}(1 - \cos\theta) \\ \Delta E &= \frac{E_0^2}{m_e c^2}(1 - \cos\theta) = \frac{(10 \times 10^3 \text{ eV})^2}{511 \times 10^3 \text{ eV}}(1 - \cos 20^\circ) = 11.8 \text{ eV}\end{aligned}$$

Thus $E_s = E_0 - \Delta E = 10,000 - 11.8 = 9988.2$ eV.

9. If the maximum energy given to an electron during Compton scattering is 20 keV, what is the wavelength of the incident photon?
Answer: The maximum wavelength shift to the photon, and thus the maximum kinetic energy transfer to the electron, is when $\theta = 180^\circ$ in which case $(1 - \cos\theta) = 2$. In that case we have the simple situation of momentum in equals momentum out, all along a straight line, or

$$\frac{E_0}{c} = \gamma\beta m_e c - \frac{E_s}{c} \qquad \text{or} \qquad E_0 = \gamma\beta m_e c^2 - E_s.$$

At the same time, we also have the conservation of energy which says

$$E_0 = E_k + E_s = (\gamma - 1)m_e c^2 + E_s.$$

Adding the right-hand form of the conservation of momentum equation with the conservation of energy equation gives

$$2E_0 = \gamma\beta m_e c^2 + \gamma m_e c^2 - m_e c^2 = m_e c^2(\gamma + \gamma\beta - 1)$$

As a result, we need to find γ and β for a 20 keV electron:

$$\gamma = 1 + \frac{E_k}{mc^2} = 1 + \frac{20}{511} = 1.0391 \qquad \text{and} \qquad \beta = \sqrt{1 - 1/\gamma^2} = 0.271$$

We then have

$$E_0 = \frac{m_e c^2}{2}(\gamma + \gamma\beta - 1) = \frac{511 \text{ keV}}{2}(1.0391 + 1.0391 \cdot 0.271 - 1) = 81.9 \text{ keV}.$$

10. A 120 keV photon undergoes Compton scattering at an angle of $50°$. Find the energy of the scattered photon, and the energy and angle of the recoil electron.
Answer: The energy of the scattered photon can be found from the Compton formula:

$$\begin{aligned} \lambda_s - \lambda_0 &= \frac{h}{m_e c}(1-\cos\theta) \\ \frac{hc}{E_s} - \frac{hc}{E_0} &= \frac{hc}{m_e c^2}(1-\cos\theta) \\ E_s &= 1/\Big[\frac{1}{E_0} + \frac{1-\cos\theta}{m_e c^2}\Big] = 1/\Big[\frac{1}{120 \text{ keV}} + \frac{1-\cos 50°}{511 \text{ keV}}\Big] = 110.7 \text{ keV} \end{aligned}$$

The energy of the scattered electron is then 120-110.7=9.3 keV, so that it has $\gamma = 1 + E_k/(m_e c^2) = 1.018$ and

$$\beta = \sqrt{1-(1+\frac{E_k}{m_e c^2})^{-2}} \simeq \sqrt{1-(1-\frac{2E_k}{m_e c^2})} \simeq \sqrt{\frac{2E_k}{m_e c^2}} = \sqrt{\frac{2 \cdot 9.3 \text{ keV}}{511 \text{ keV}}}$$

or $\beta \simeq 0.191$ so that the electron's net momentum is

$$p_e = \gamma\beta m_e c^2 = (1.018)(0.191)(511 \text{ keV}/c^2)c = 99.4 \text{ keV}/c.$$

Finally, we can get the angle φ of the scattered electron setting the $\hat{y}$ momenta to be equal and opposite:

$$\begin{aligned} (E_s/c)\sin\theta &= p_e \sin\varphi \\ \varphi &= \arcsin\Big[\frac{E_s/c}{p_e}\sin\theta\Big] = \arcsin\Big[\frac{110.7 \text{ keV}/c}{99.4 \text{ keV}/c}\sin 50°\Big] = 58.6° \end{aligned}$$