PHY 251 Fall 2009: homework problem set 2, due PHY 251 drop box in room A-129 by noon on Friday, Sep. 18.

1. Dick and Jane are each carrying an exact meter stick (according to them in their frame of reference). Dick goes past you at mach ten or 3000 m/s, while Jane goes past you at half the speed of light. How much shorter than 1 meter does Jane's meter stick appear to you? How much shorter than 1 meter does Dick's meter stick look to you?

   Answer: In both cases, you see a length $\ell = \ell_0/\gamma$ relative to what they see, or a length difference of $\ell_0 - \ell = \ell_0(1 - 1/\gamma)$. In the case of Jane, we have $\beta = 0.5$ and $\gamma = 1/\sqrt{1-(1/2)^2} = 2/\sqrt{3}$ giving

   $$\ell_0 - \ell = \ell_0(1 - \frac{1}{\gamma}) = (1 \text{ m})(1 - \frac{\sqrt{3}}{2}) = 0.134 \text{ meters.}$$

   In the case of Dick, we have $\beta = (v/c) = (3 \times 10^3 / 3 \times 10^8) = 10^{-5}$ so we have to use a low-β expansion for the Lorentz factor γ:

   $$\begin{aligned} \ell_0 - \ell &= \ell_0(1 - \frac{1}{\gamma}) = \ell_0\left(1 - (1-\beta^2)^{1/2}\right) \\ &\simeq \ell_0\left(1 - (1 - \frac{1}{2}\beta^2)\right) = \ell_0 \frac{1}{2}\beta^2 \\ &= (1 \text{ m})\frac{1}{2}(10^{-5})^2 = 5 \times 10^{-10} \text{ m.} \end{aligned}$$

   Given that atoms have a size of around 2×10^{-10} m, this is going to be hard to detect!

2. A smug city slicker bets a farmer that he can't get his 10 m long ladder into a 8 m long shed. The farmer, who reads Einstein each day after milking his cows, takes him up on the bet. He tells the city slicker to stand to the side of the shed and look in the windows at each end, and the farmer then runs fast as he can through the shed while carrying the ladder. How fast does the farmer have to run to win the bet? While on the run, how long does the shed appear to the farmer, and does the farmer ever think his ladder is entirely inside the shed? (3 answers required).

   Answer: For the city slicker in the stationary frame S_1 to observe the ladder as being completely inside the shed while the farmer runs by in the farmer's frame S_2, we must have a Lorentz contraction calculated from $L_1 = (1/\gamma)L_2$. Therefore

   $$\gamma = \frac{L_2}{L_1} = \frac{10 \text{ m}}{8 \text{ m}} = \frac{5}{4}.$$

   Now $\gamma^2 = 1/(1-\beta^2)$ so

   $$\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 4^2/5^2} = \sqrt{9/25} = 3/5$$

   or $v = 0.60c$. Now, to the farmer the shed appears to be moving toward him at $v = 0.60c$, so its length is Lorentz contracted according to $\gamma = 5/4$, giving an apparent shed length of $(8 \text{ m})/(5/4) = 6.4$ m. As a result, the farmer never sees the ladder as being completely inside the shed! But the farmer knows what the city slicker sees, and collects on the bet...

3. You tune your radio in your car to your favorite radio station which plays all Barry Manilow all the time: 88.1 MHz. However, you find instead that you are receiving the death metal station WIKD which broadcasts at 106.1 MHz. How fast are you going, and in which direction relative to WIKD?

   Answer: The source is emitting at $\nu_0 = 106.1$ MHz because you're hearing to WIKD even though you set your radio to 88.1 MHz. Therefore the frequency you receive is red-shifted ($\nu = 88.1$ MHz) which means you are traveling away from the source (or the source is receding from you), so that this is a Doppler shift problem with $\theta = 0°$ and $\cos\theta = 1$ giving

$$\begin{aligned}
\nu &= \frac{\nu_0}{\gamma(1+\beta)} = \nu_0\frac{\sqrt{1-\beta^2}}{1+\beta} = \nu_0\frac{\sqrt{(1+\beta)(1-\beta)}}{1+\beta} = \nu_0\sqrt{\frac{1-\beta}{1+\beta}} \\
(\frac{\nu}{\nu_0})^2 &= \frac{1-\beta}{1+\beta} \\
(\frac{\nu}{\nu_0})^2(1+\beta) &= 1-\beta \\
\beta\left(1+(\frac{\nu}{\nu_0})^2\right) &= 1-(\frac{\nu}{\nu_0})^2 \\
\beta &= \frac{1-(\nu/\nu_0)^2}{1+(\nu/\nu_0)^2} = \frac{1-(88.1/106.1)^2}{1+(88.1/106.1)^2} = 0.184
\end{aligned}$$

   so the person is traveling at 0.184 times the speed of light, away from station WIKD.

4. Which carries more energy: a kilogram of gasoline, a kilogram of TNT or dynamite (which releases about 15 MJ per kg), or a 200 micrometer diameter drop of water if all of its mass could be converted into energy? (Hint: use Wikipedia to remind yourself of the heat of combustion).

   Answer: From wikipedia, the heat of combustion of gasoline is about 44 MJ/kg (using the LHV value). Gasoline carries *more* energy per mass than TNT does; it's just that TNT "burns" so quickly that we call it an explosion. A 1 micrometer diameter drop of water has a mass of

$$m = V\rho = \frac{4}{3}\pi(100\times10^{-6}\text{ meters})^3\cdot(\frac{1\text{ g}}{\text{cm}^3})\cdot(\frac{10^2\text{ cm}}{\text{m}})^3\cdot(\frac{\text{kg}}{10^3\text{ g}}) = 4.2\times10^{-9}\text{ kg}$$

   so it carries a $E = mc^2$ energy equivalent of

$$E = mc^2 = (4.2\times10^{-9}\text{ kg})\cdot(3\times10^8\text{ m/s})^2 = 3.8\times10^8\text{ Joules} = 380\text{ MJ}$$

   so the water drop carries the most energy if one could turn all the energy into mass. . .

5. In a quest to win either a Darwin Award or a Nobel Prize, you decide that you want to build a particle accelerator in your dorm. You use unshielded terminals to hook up an accelerating voltage of 200 kiloVolts, and also build a 0.2 Telsa magnet using soldered paper clips for the windings. If it is an electron that you accelerate, what is its kinetic energy? Momentum? Radius of curvature in the magnetic field?

Answer: The energy given to an electron is qV so its kinetic energy is 200 keV and the rest mass of an electron is 511 keV/c^2 or 9.11×10^{-31} kg. We can find the momentum from

$$\begin{aligned}
E_k &= (\gamma - 1)mc^2 \qquad \Rightarrow \qquad \gamma = 1 + \frac{E_k}{mc^2} = 1 + \frac{200 \text{ keV}}{511 \text{ keV}} = 1.391 \\
\gamma &= (1 - \beta^2)^{-1/2} \qquad \Rightarrow \qquad 1 - \beta^2 = 1/\gamma^2 \\
\Rightarrow \qquad \beta &= \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 1/(1.391^2)} = 0.695 \\
p &= \gamma m v = \gamma \beta m c^2/c = (1.391)(0.695)(511 \text{ keV})/c = 494 \text{ keV}/c.
\end{aligned}$$

The radius of curvature is found from setting the Lorentz force equal to the centripetal force needed to maintain uniform circular motion:

$$\begin{aligned}
qvB &= \gamma m \frac{v^2}{r} \\
r &= \frac{\gamma m v^2}{qvB} = \frac{\gamma m v}{qB} \\
&= \frac{\gamma m \beta c}{qB} = \frac{(1.39)(9.11 \times 10^{-31} \text{ kg})(0.695)(3.00 \times 10^8 \text{ m/s})}{(1.602 \times 10^{-19} \text{ C})(0.2 \text{ T})} = 8.24 \times 10^{-3} \text{ m}
\end{aligned}$$

or 8.24 mm.

6. The LHC at CERN will accelerate protons up to an energy of 7 TeV, or 7×10^{12} eV. Calculate the momentum of such a proton. How different from the speed of light is its velocity?
Answer: Since protons have a mass of 0.9383 GeV/c^2, and the protons receive a kinetic energy of 7,000 GeV, the Lorentz factor is $\gamma = 1 + E/E_0 = 1 + 7000/.9383 = 7461$. This means that β will be very near 1, so let's say that $\beta = 1 - \epsilon$ where ϵ is small. We then have

$$\begin{aligned}
\gamma &= [1 - (1 - \epsilon)^2]^{-1/2} = [1 - (1 - 2\epsilon + \epsilon^2)]^{-1/2} \simeq [2\epsilon]^{-1/2} \\
\text{Thus } \epsilon &= 1/(2\gamma^2) = 1/(2 \cdot 7461^2) = 9.0 \times 10^{-9}
\end{aligned}$$

so $\beta = 1 - 9.0 \times 10^{-9}$, or very nearly 1. Now for momentum, we know that $p = \gamma m v = \gamma m \beta c$ but since β is so nearly 1 we can say that

$$p = \gamma m c = 7461 \cdot (0.9383 \text{ GeV}/c^2) \cdot c = 7000 \text{ GeV}/c = 7 \text{ TeV}/c.$$

That is, when γ is very large we approach the case where $p \simeq E/c$ (and remember that for photons $p = E/c$ is exact).

7. Radiation pressure can be described using the momentum of a photon. Use conservation of momentum to derive an expression for the acceleration a of a mass m that has absorbed N photons of energy $E = h\nu$ each over a time t.
Answer: I think it's better to write "per time" as Δt instead of t, and the increment of photons as ΔN. Let's do so. Beforehand we have the object with mass m at rest, and an incident photon with energy $E = h\nu$ and momentum $p = E/c = h\nu/c$. Afterwards the photon has been absorbed so that it has no momentum, and the mass has received all of the momentum. Because $F = ma = dp/dt$, we can say that the mass receives an acceleration of

$$a = \frac{dp/dt}{m} = \frac{\Delta p}{\Delta t\, m} = \frac{h\nu/c}{\Delta t\, m}$$

per photon, or an acceleration of

$$a = \frac{h\nu}{mc} \cdot \frac{\Delta N}{\Delta t} = \frac{h}{m\lambda} \cdot \frac{\Delta N}{\Delta t}$$

for a flux of ΔN photons per time interval Δt. By the way, for a perfect reflector the acceleration is nearly twice that. Why nearly? Because the photon that bounces back is red-shifted due to the energy transfer to the mass. The red shift is very very small and can usually be ignored.

8. Use your results from the previous problem to calculate the acceleration you might experience in a solar sailing race. Let's say that you, your space "shell," and your absorptive solar sail together weigh 1000 kg. Let's assume that your solar sail is a square 2 km on a side, that it's absorptive, that it's facing dead-on to the sun, and that the sun's output of 1366 W/m^2 is all visible light ($\lambda = 500$ nm).
Answer: We know the acceleration for a given number of photons per time. If the Sun's output is all at $\lambda = 500$ nm with an energy per photon of $E = h\nu = hc/\lambda$ with $hc = 1240$ eV·nm, we can convert the power per square meter P times the sail area A, along with the $E = hc/\lambda$ energy per photons to find the number of photons per second $\Delta N/\Delta t$ on the sail as

$$\begin{aligned} \frac{\Delta N}{\Delta t} &= PA\frac{\lambda}{hc} \\ &= (1366\ \frac{\text{Joules}}{\text{m}^2 \cdot \text{sec}}) \cdot (2000\ \text{m})^2 \cdot (\frac{1\ \text{photon}}{(1240\ \text{eV} \cdot \text{nm})/(500\ \text{nm})}) \cdot (\frac{1\ \text{eV}}{1.6 \times 10^{-19}\ \text{Joules}}) \end{aligned}$$

or $\Delta N/\Delta t = 1.4 \times 10^{28}$ photons per second. Now use this in our earlier result for acceleration:

$$\begin{aligned} a &= \frac{h}{m\lambda} \cdot \frac{\Delta N}{\Delta t} = \frac{h}{m\lambda} \cdot PA\frac{\lambda}{hc} = \frac{PA}{mc} \\ &= \frac{6.63 \times 10^{-34}\ \text{Joules} \cdot \text{sec}}{(1000\ \text{kg}) \cdot (500 \times 10^{-9}\ \text{meters})} \cdot (1.4 \times 10^{28}\ \frac{\text{photons}}{\text{sec}}) = 0.019\ \frac{\text{meters}}{\text{sec}^2} \end{aligned}$$

which is a very small, but non-zero, acceleration.

9. Calculate the peak output wavelength for an object heated to a temperature of 10^5 K.
Answer: The peak wavelength is

$$\lambda_{\text{peak}} = \frac{hc}{4.965 k_B T} = \frac{1240\ \text{eV} \cdot \text{nm}}{4.965 \cdot (8.62 \times 10^{-5}\ \text{eV/K}) \cdot (10^5\ \text{K})} = 28.9\ \text{nm}$$

which is in the ultraviolet.

10. Calculate the number of photons per second emitted by a low power Bluetooth radio transmitter (1 mW at 2.4 GHz).
Answer: The energy per photon is

$$E = h\nu = (6.63 \times 10^{-34}\ \text{Joule} \cdot \text{sec}) \cdot (2.4 \times 10^9\ \text{sec}^{-1}) = 1.6 \times 10^{-24}\ \text{Joules/photon}.$$

The number of photons per second is then

$$\frac{10^{-3}\ \text{Joules/sec}}{1.6 \times 10^{-24}\ \text{Joules/photon}} = 6.3 \times 10^{20}\ \text{photons/sec}$$

so it's no wonder that nobody ever noticed individual photons coming out of radio transmitters. . .