

PHY 251 Fall 2009: homework problem set 1, due PHY 251 drop box in room A-129 by noon on Friday, Sep. 11.

Some general comments: you'll see that I always work out an algebraic solution first and then plug in the numbers at the end. Also, if you put mks (meter · kilogram · second) units in, you get an answer in mks units out. Therefore I tend to be lazy about writing the units in the numerical calculations.

1. Show that if one uses the Galilean relativity transformation of

$$\begin{aligned}x_2 &= x_1 - vt_1 \\y_1 &= y_2 \\z_1 &= z_2 \\t_1 &= t_2\end{aligned}\tag{1}$$

and the relationships for expansion of light spheres of

$$x_1^2 + y_1^2 + z_1^2 - c^2 t_1^2 = 0\tag{2}$$

$$x_2^2 + y_2^2 + z_2^2 - c^2 t_2^2 = 0.\tag{3}$$

that you get a non-general and inconsistent-with-classical-physics result.

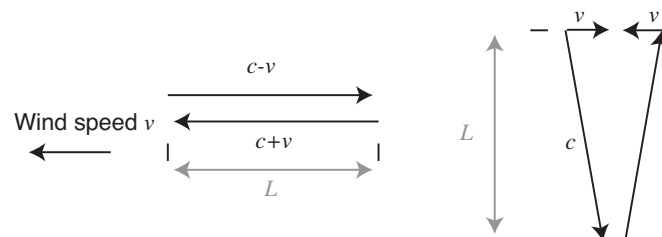
Answer: Let's do the substitutions in the second equation:

$$\begin{aligned}(x_1 - vt_1)^2 + y_1^2 + z_1^2 - c^2 t_1^2 &= 0 \\x_1^2 + y_1^2 + z_1^2 - c^2 t_1^2 + v^2 t_1^2 - 2vx_1 t_1 &= 0 \\ \text{But } x_1^2 + y_1^2 + z_1^2 - c^2 t_1^2 = 0 \quad \text{so } v^2 t_1^2 - 2vx_1 t_1 &= 0 \\ &\text{giving } v = 2\frac{x_1}{t_1}\end{aligned}$$

which implies that Galilean relativity works only at a particular relationship between distance and time which differs by a factor of 2 from the usual $v = \Delta x / \Delta t$.

2. Show how classical relativity affects speed in an airplane flight with either headwinds and tailwinds, or crosswinds. Let's say that the wind is blowing east west at a velocity of $v = 10$ m/s, and that you're in an airplane that travels at a speed of $c = 250$ m/s. Consider a round-trip journey where you fly 300 km out and back as measured on the ground, going east then west. Then consider the case where you go straight south then north as measured from the ground (what compass heading should you fly the plane at to accomplish this?). What's the time difference between the two trips? Work out an algebraic answer approximated to lowest order in $\beta = v/c$ before you plug in numbers.

Answer: Here's a diagram:



When the airplane is flying against and with the wind, its net travel time $t = \text{distance}/\text{velocity}$ is

$$t_1 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{L}{c} \left(\frac{1}{1-\beta} + \frac{1}{1+\beta} \right)$$

with $\beta \equiv v/c$. Now for small x the binomial expansion says $(1+x)^r \simeq 1 + rx + \frac{1}{2}r(r-1)x^2 + \dots$ so we have

$$\begin{aligned} t_1 &= \frac{L}{c} \left((1-\beta)^{-1} + (1+\beta)^{-1} \right) \\ &\simeq \frac{L}{c} \left(1 + (-1)(-\beta) + \frac{1}{2}(-1)(-1-1)(-\beta)^2 + \dots + 1 + (-1)(\beta) + \frac{1}{2}(-1)(-1-1)(\beta)^2 + \dots \right) \\ &\simeq \frac{L}{c} \left(1 + \beta + \beta^2 + \dots + 1 - \beta + \beta^2 + \dots \right) = 2\frac{L}{c} \left(1 + \beta^2 \right). \end{aligned}$$

When the plane is traveling across the wind, it has to fly along a compass bearing of $\tan \theta = v/c$ (or $\theta = \tan^{-1}(10/250) = 2.3^\circ$ in this case) to travel due north or south as seen on ground. Its speed c' along the straight line distance from start to destination is found from $c^2 = c'^2 + v^2$ which gives

$$c' = \sqrt{c^2 - v^2} = c\sqrt{1 - \beta^2}.$$

This applies the same way to both cases. The travel time is then

$$\begin{aligned} t_2 &= \frac{L}{c} \left(\frac{1}{\sqrt{1-\beta^2}} + \frac{1}{\sqrt{1-\beta^2}} \right) = 2\frac{L}{c} \left((1-\beta^2)^{-1/2} \right) \\ &= 2\frac{L}{c} \left(1 + \frac{1}{2}\beta^2 + \mathcal{O}(\beta^4) \right) \end{aligned}$$

where $\mathcal{O}(\beta^4)$ means that the next additional terms are of order β^4 which are very small if β is small. We can then subtract the two times:

$$\Delta t = t_1 - t_2 = 2\frac{L}{c} \left(1 + \beta^2 \right) - 2\frac{L}{c} \left(1 + \frac{1}{2}\beta^2 \right) = 2\frac{L}{c} \frac{1}{2}\beta^2 = \frac{L}{c}\beta^2$$

With $L = 300$ km, $c = 250$ m/s, and $v = 10$ m/s, we have

$$\Delta t = \frac{300 \times 10^3 \text{ m}}{250 \text{ m/s}} \left(\frac{10 \text{ m/s}}{250 \text{ m/s}} \right)^2 = 1.9 \text{ seconds}$$

3. How fast must a meter stick be moving past you for it to appear to you to be only 0.6 m long?

Answer: In $\ell' = \ell_0/\gamma$, we solve for γ :

$$\gamma = \frac{\ell_0}{\ell'} = \frac{1.0 \text{ m}}{0.6 \text{ m}} = \frac{5}{3}.$$

We then want to find β from γ :

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\beta^2}} \\ \frac{1}{\gamma^2} &= 1-\beta^2 \\ \beta^2 &= 1-\frac{1}{\gamma^2} \\ \beta &= \sqrt{1-1/\gamma^2} = \sqrt{1-\frac{1}{(5/3)^2}} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}\end{aligned}$$

so we have $v = 0.8c$.

4. It is said that “time flies when you’re having fun.” Some people having fun go past you at $v = 3 \times 10^6$ m/s. You see one of them laugh for 20 seconds straight as observed in your frame. How much different is the duration of their laugh in their frame?

Answer: We see time dilated by γ relative to the moving frame; we see $t' = 20$ seconds while those who are having fun see a time of $t_0 = t'/\gamma$ which, since $\gamma \geq 1$, is less than t' . The time difference is then

$$\begin{aligned}\Delta t &= t' - t_0 = t' - \frac{t'}{\gamma} = t'[1 - (1 - \beta^2)^{1/2}] \simeq t'[1 - (1 - \frac{1}{2}\beta^2)] \\ &\simeq t'\frac{1}{2}\beta^2 = (20 \text{ sec})\frac{1}{2}\left(\frac{3 \times 10^6 \text{ m/s}}{3 \times 10^8 \text{ m/s}}\right)^2 = (20 \text{ sec})\frac{1}{2}10^{-4} = 0.001 \text{ sec}.\end{aligned}$$

5. While staring out the window, you are somewhat astonished to see a spear go flying past, and you snap a photograph. From your photo you see that it is 1 meter long and tilted up at an angle of 20° from the x direction, and it goes flying past at $v_x = 1 \times 10^8$ m/sec and $v_y = 0$. What is length, and tilt angle, of the spear in its own inertial reference frame?

Answer: Because of relative motion in the x direction, we see a contracted version of the spear’s length. There’s no motion in the y direction, so we see the true length in that direction. That is, we see $x' = x_0/\gamma$ and $y' = y_0$ and from that we can say $r' = \sqrt{x'^2 + y'^2}$ and $\tan(\theta') = y'/x'$. Note that

$$\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\left(\frac{1 \times 10^8}{3 \times 10^8}\right)^2} = 1/\sqrt{1-\left(\frac{1}{3}\right)^2} = 1/\sqrt{8/9} = 3/\sqrt{8} = 1.061.$$

We can relate these to dimensions in the spear’s frame of reference as follows:

$$\begin{aligned}r_0 &= \sqrt{x_0^2 + y_0^2} = \sqrt{x'^2\gamma^2 + y'^2} \\ &= \sqrt{[1 \cos(20^\circ)]^2(1.061)^2 + [1 \sin(20^\circ)]^2} = 1.054 \text{ meters} \\ \tan \theta_0 &= \frac{y_0}{x_0} = \frac{y'}{\gamma x'} = \frac{1}{\gamma} \tan \theta' \\ \text{or } \theta_0 &= \tan^{-1}[\tan(\theta')/\gamma] = \tan^{-1}[\tan(20^\circ)/(1.061)] = 18.9^\circ.\end{aligned}$$

That is, the meter stick is longer than 1 meter, and is rotated by a smaller angle when viewed in its own frame.

6. A cop pulls you over for running a red light ($\lambda = 650 \text{ nm}$). Thinking that you're a clever physics student, you go to court and point out that due to the relativistic Doppler shift it looked like a green ($\lambda = 550 \text{ nm}$) light to you, smiling smugly at the judge. However, the judge immediately slams you with a speeding ticket instead. How fast does the judge know you were going?

Answer: For a source moving straight towards the observer, the relativistic Doppler shift is (Serway Eq. 1.15),

$$\begin{aligned}\nu' &= \nu_0 \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \\ \frac{c}{\lambda'} &= \frac{c}{\lambda_0} \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \\ \left(\frac{\lambda_0}{\lambda'}\right)^2 &= \frac{1+\beta}{1-\beta}\end{aligned}$$

If we define $A \equiv (\lambda_0/\lambda')^2$, we have

$$\begin{aligned}A &= \frac{1+\beta}{1-\beta} \\ A - A\beta &= 1 + \beta \\ A - 1 &= (A + 1)\beta \\ \text{giving } \beta &= \frac{A - 1}{A + 1} = \frac{(\lambda_0^2 - \lambda'^2)/\lambda'^2}{(\lambda_0^2 + \lambda'^2)/\lambda'^2} \\ \text{or } \beta &= \frac{\lambda_0^2 - \lambda'^2}{\lambda_0^2 + \lambda'^2}.\end{aligned}$$

Since $\lambda_0 = 650 \text{ nm}$ (it's really a red light) and $\lambda' = 550 \text{ nm}$ (we think it's a green light), we come up with $\beta = 0.166$ or $v = \beta c = 4.95 \times 10^7 \text{ m/s}$ or

$$(4.9 \times 10^7 \frac{\text{m}}{\text{s}}) \cdot \left(\frac{1 \text{ inch}}{0.0254 \text{ m}}\right) \cdot \left(\frac{1 \text{ foot}}{12 \text{ inches}}\right) \cdot \left(\frac{1 \text{ mile}}{5280 \text{ feet}}\right) = 3.07 \times 10^5 \frac{\text{miles}}{\text{hour}}$$

A bit too fast, don't you think?

7. For $v \ll c$, show that the relativistic Doppler shift leads to relative frequency and wavelength shifts of

$$\frac{\Delta\nu}{\nu} \simeq -\frac{v}{c} \quad \text{and} \quad \frac{\Delta\lambda}{\lambda} \simeq \frac{v}{c}.$$

Apply this to find the velocity at which a galaxy must be receding if we see a spectral feature at $\lambda = 400 \text{ nm}$ redshifted by 15 nm .

Answer: From Serway Eq. 1.15, we have

$$\nu' = \nu_0 \frac{(1-\beta)^{1/2}}{(1+\beta)^{1/2}} = \nu_0 (1-\beta)^{1/2} (1+\beta)^{-1/2} \simeq \nu_0 \left(1 - \frac{1}{2}\beta\right) \left(1 - \frac{1}{2}\beta\right) \simeq \nu_0 (1-\beta)$$

We can then find

$$\frac{\Delta f}{f} = \frac{\nu' - \nu_0}{\nu_0} = \frac{\nu_0(1 - \beta) - \nu_0}{\nu_0} = \frac{-\beta\nu_0}{\nu_0} = -\beta.$$

Since $\lambda = c/\nu$, we have

$$\begin{aligned} d\lambda &= d\left(\frac{c}{\nu}\right) = -c\frac{d\nu}{\nu^2} \\ \frac{\nu}{c}d\lambda &= -\frac{d\nu}{\nu} \\ \frac{d\lambda}{\lambda} &= -\frac{d\nu}{\nu} \end{aligned}$$

so

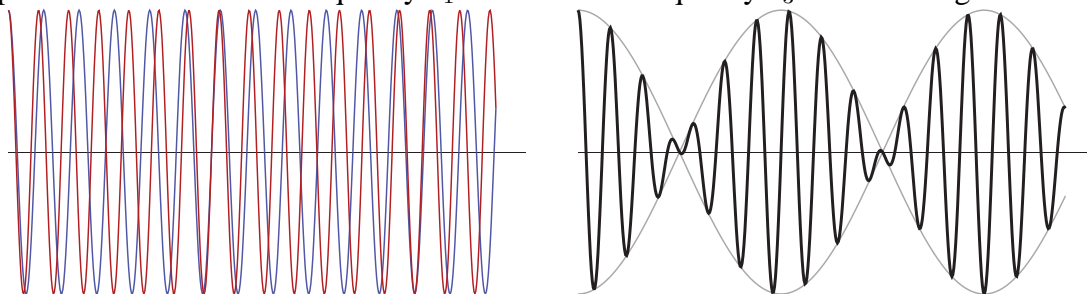
$$\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\nu}{\nu} \simeq -(-\beta) = \beta$$

Now if $\Delta\lambda = 15 \text{ nm}$ and $\lambda = 400 \text{ nm}$, we have

$$\beta = \frac{\Delta\lambda}{\lambda} = \frac{15}{400} = 0.0375$$

or $v = 0.0375c = 1.13 \times 10^7 \text{ m/sec}$.

8. A radar speed gun works by sending out a radar wave at a frequency ν_1 . A moving object sees this frequency shifted to ν_2 , and reflects that radar signal back to the radar gun. From the radar gun's point of view, we have a moving emitter at frequency ν_2 which leads to a frequency ν_3 seen by the radar gun. We then get a beat frequency $\nu_b = \nu_3 - \nu_1$ from the superposition of the emitted frequency ν_1 and received frequency ν_3 at the radar gun:



Show that, to lowest order in (v/c) , the beat frequency is given by $\nu_b = 2(v/c)\nu_1$, and calculate the beat frequency the radar gun sees when measuring an object moving at $v = 30 \text{ m/sec}$ using a radar frequency of 10.0 GHz .

Answer: There are *two* Doppler shifts here! The radar transmitter puts out a signal at a frequency ν_1 , but the object which is moving towards the transmitter ($\theta = \pi$) sees a Doppler-shifted frequency ν_2 of

$$\nu_2 = \frac{\nu_1}{\gamma(1 + \beta \cos(\pi))} = \nu_1 \frac{\sqrt{1 - \beta^2}}{1 - \beta} = \nu_1 \frac{\sqrt{1 - \beta} \sqrt{1 + \beta}}{1 - \beta} = \nu_1 \frac{\sqrt{1 + \beta}}{\sqrt{1 - \beta}}.$$

The object then reflects this signal so that it is a signal emitter at frequency ν_2 . Since it's moving towards the radar gun, the radar gun sees a frequency ν_3 which is the Doppler-shifted version of ν_2 for which we have the same form of Doppler shift for $\theta = \pi$:

$$\nu_3 = \nu_2 \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} = \nu_1 \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} = \nu_1 \frac{1+\beta}{1-\beta} = \nu_1 \frac{v+c}{v-c}.$$

In this case, the beat frequency ν_b is

$$\begin{aligned} \nu_b &= \nu_3 - \nu_1 = \nu_1 \frac{1+\beta}{1-\beta} - \nu_1 = \nu_1 (1+\beta)^1 (1-\beta)^{-1} - \nu_1 \\ &\simeq \nu_1 (1+\beta)(1+\beta) - \nu_1 \simeq \nu_1 (1+2\beta) - \nu_1 = 2\beta\nu_1 \end{aligned}$$

where we have assumed $\beta \ll 1$. For an object with a speed of $v = 30.0$ m/s and a microwave frequency of $\nu_1 = 10.0$ GHz, the beat frequency is

$$\nu_b = 2\beta\nu_0 = 2 \frac{30.0}{2.99 \times 10^8} \cdot (10.0 \times 10^9) = 2.00 \times 10^3$$

or 2 kHz which is easily measured by cheap, low frequency electronics. If the measurement of the beat frequency is good to ± 5 Hz, then we have measured the speed to an accuracy of $(5/2000) = 0.0025$ or 0.25% and $0.0025 \cdot 30.0 = 0.075$ miles per hour. You're not likely to argue your way out of a speeding ticket by claiming that the radar gun is highly inaccurate!

9. As a still-lurid 80-year old in 2061, Paris Hilton drives a Hummer H34 spaceship which is 15 m long. Al Gore, long since retired but still kicking around, drives a Honda CivicLesson spaceship which is only 3 m long. Paris and Al both fly past you while you're standing on earth, and it looks to you like they're both driving spaceships of the same length. You know that Al drives at, but not even a smidgen above, the posted speed limit of $v = 0.55c$. How fast is Paris going relative to you on earth? Relative to Al?

Answer: In their respective frames, Paris' Hummer has a proper length of $L_{1,0} = 15$ m while Al's CivicLesson has proper length of $L_{2,0} = 3$ m. We also see a speed of $\beta_2 = 0.55$ for Al (giving $\gamma_2 = 1.20$). For both vehicles, we see the same length in our frame, or

$$\begin{aligned} L'_1 &= L'_2 \quad \text{giving} \quad \frac{L_{1,0}}{\gamma_1} = \frac{L_{2,0}}{\gamma_2} \\ \gamma_1 &= \frac{L_{1,0}}{L_{2,0}} \gamma_2 = \frac{15}{3} \cdot 1.20 = 6.0. \end{aligned}$$

Now from $\gamma = 1/\sqrt{1-\beta^2}$ we can find $\beta = \sqrt{1-1/\gamma^2}$ so Paris' speed relative to us is

$$\beta_1 = \sqrt{1-1/(6.0)^2} = 0.986$$

as viewed by us on the ground. We now need to shift by a velocity of $0.55c$ from our frame to Al's frame to see how Al perceives the speed of Paris' Hummer H34, so that we have

$v_1 = 0.986c$ (what we see for Paris) and $v = 0.55c$ (the velocity needed to shift into Al's frame), or

$$v_2 = \frac{v_1 - v}{1 - vv_1/c^2} = \frac{0.986c - 0.55c}{1 - 0.55 \cdot 0.986} = 0.953c.$$

10. The nearest star to earth is Proxima Centauri, at 4.2 light years distance. If you were instantaneously able to accelerate to $\beta = 0.1$, how long would the journey appear to take to you in your spaceship? Next, using classical physics, estimate how long it would take for you to accelerate to $\beta = 0.1$ if your acceleration was limited to $2.0g$.

Answer: To an observer stationary on earth, your travel time would be $t' = (4.2 \text{ light speed} \cdot \text{years}) / (0.1 \text{ light speed}) = 42 \text{ years}$. But the stationary observer sees a time t' dilated compared to what you see in the spacecraft; you see $t_0 = t' / \gamma$ or

$$t_0 = \frac{42 \text{ years}}{1/\sqrt{1 - (0.1)^2}} = 41.8 \text{ years}.$$

In classical physics, $\Delta v = a(\Delta t)$ or

$$\begin{aligned} \Delta t &= \frac{\Delta v}{a} = \frac{0.1 \cdot 3 \times 10^8 \text{ m/s}}{2 \cdot 9.8 \text{ m/s}^2} = 1.53 \times 10^6 \text{ sec} \\ &= (1.53 \times 10^6 \text{ sec}) \cdot \frac{1 \text{ year}}{3600 \cdot 24 \cdot 365 \text{ sec}} = 0.05 \text{ years}. \end{aligned}$$

That is, you have to suffer 0.05 years or 2.5 weeks of hard acceleration to have a relativistic gain of 0.2 years during the journey. But even with relativity accounted for it would take an enormous length of time to get there. In the next problem set we'll estimate the mass of rocket fuel we would need to make the trip. . .