PHY 251 Fall 2009: Final Exam (Thursday, Dec. 17). Some masses in u:

$^{1}_{1}$H	1.007 825	$^{206}_{82}$Pb	205.974 449	$^{210}_{84}$Po	209.982 857
$^{2}_{1}$H	2.014 102	$^{208}_{83}$Bi	207.979 727	$^{211}_{84}$Po	210.986 637
$^{3}_{2}$He	3.016 029	$^{209}_{83}$Bi	208.980 383	$^{212}_{85}$At	211.990 735
$^{4}_{2}$He	4.002 603	$^{209}_{84}$Po	208.982 416	$^{214}_{86}$Rn	213.995 346

1. Calculate the wavelengths associated with a 100 eV photon, a 20 eV electron, and a 200 keV electron.
Answer: The wavelength of the 100 eV photon is

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{100 \text{ eV}} = 12.4 \text{ nm}.$$

The 20 eV electron is non-relativistic, so we have $E_k = p^2/2m$ and $p = \sqrt{2mE_k}$ giving a wavelength of

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}} = \frac{hc}{\sqrt{2mc^2E_k}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2 \cdot (511 \times 10^3 \text{ eV}) \cdot (20 \text{ eV})}} = 0.274 \text{ nm}.$$

The 200 keV electron energy is close enough to the rest mass that we must use a relativistic approach. We know $E_k = (\gamma - 1)mc^2$ so

$$\gamma = 1 + \frac{E_k}{mc^2} = 1 + \frac{200 \text{ keV}}{511 \text{ keV}} = 1.39$$

and then from $\gamma \equiv 1/\sqrt{1-\beta^2}$ we can find

$$\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 1/(1.39^2)} = 0.695$$

We can then use $\lambda = h/p$ and $p = \gamma m v = \gamma\beta mc$ to find

$$\lambda = \frac{h}{\gamma\beta mc} = \frac{hc}{\gamma\beta mc^2} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.39 \cdot 0.695 \cdot (511 \times 10^3 \text{ eV})} = 0.0025 \text{ nm}.$$

2. A He^{+} ion has its electron undergo a transition from the $2p$ to the $1s$ state. What is the mean wavelength of the emitted photon? If the $2p$ state were to have a lifetime of 2×10^{-14} seconds or 20 fsec, what would be the wavelength spread about the mean?
Answer: We have $E = -E_0 Z^2/n^2$ and a correction of m_r/m_e with

$$\frac{m_r}{m_e} = \frac{(4 \cdot 939 \cdot 0.511)/(4 \cdot 939 + 0.511)}{0.511} = 0.99986$$

which we will ignore. The photon energy is then

$$\Delta E = E_2 - E_1 = (-13.6)\frac{2^2}{2^2} - (-13.6)\frac{2^2}{1^2} = 13.6(4-1) = 40.8 \text{ eV}$$

and the photon wavelength is $\lambda = hc/E = 1240/40.8 = 30.4$ nm. The energy width is given by the uncertainty principle $(\Delta E)(\Delta t) = \hbar/2$ or

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{6.582 \times 10^{-16} \text{ eV} \cdot \text{sec}}{2 \cdot (2 \times 10^{-14} \text{ sec})} = 0.016 \text{ eV}$$

and since $\Delta E/E = \Delta\lambda/\lambda$ we have

$$\Delta\lambda = \lambda\frac{\Delta E}{E} = \frac{hc}{E}\frac{\Delta E}{E} = \frac{(1240\text{ eV}\cdot\text{nm})(0.016\text{ eV})}{(40.8\text{ eV})^2} = 0.012\text{ nm}$$

as the 1σ spread in wavelength.
Note: it is incorrect to say $\Delta\lambda = hc/\Delta E$! Instead:

$$\lambda = \frac{hc}{E} \qquad \rightarrow \qquad d\lambda = hc\frac{-1}{E^2}dE = -\frac{hc}{E}\frac{dE}{E} = -\lambda\frac{dE}{E}$$

so $\Delta\lambda/\lambda = -\Delta E/E$ and of course $|\Delta\lambda|/\lambda = |\Delta E|/E$ in terms of relating energy and wavelength spreads.

3. An electron curves with a radius of 6 mm in a magnetic field of 0.2 T. Calculate the speed, kinetic energy, total energy, and momentum of the electron.
Answer: We have

$$\begin{aligned} qvB &= \gamma m\frac{v^2}{r} \\ \gamma\beta &= \frac{qBr}{mc} = \frac{(1.6\times10^{-19})\cdot(0.2)\cdot(0.006)}{(9.11\times10^{-31})\cdot(3\times10^8)} = 0.703 \qquad (1) \end{aligned}$$

Given $\gamma\beta$, we can find β:

$$\begin{aligned} \text{Let } x &= \gamma\beta \\ x^2 &= \frac{\beta^2}{1-\beta^2} \\ x^2 - x^2\beta^2 &= \beta^2 \\ \beta^2(1+x^2) &= x^2 \\ \beta &= \frac{x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+1/(\gamma\beta)^2}} \end{aligned}$$

Therefore

$$\beta = \frac{1}{\sqrt{1+1/(\gamma\beta)^2}} = \frac{1}{\sqrt{1+1/(0.703)^2}} = 0.575$$

and $\gamma = 1/\sqrt{1-\beta^2} = 1.22$. The total energy is $\gamma mc^2 = 1.22\cdot 511$ keV or 623 keV, the kinetic energy is $(\gamma-1)mc^2 = 112$ keV, and the momentum is

$$p = \gamma mv = \gamma\beta mc = 0.703\cdot(511\text{ keV}/c^2)\cdot c = 359\text{ keV}/c.$$

4. An electron placed in an infinite square well has an energy of 2.0 eV when in its ground state. How wide is the well? How much energy must be added to excite the electron to the next available state?
Answer: The energy states of an infinite square well potential are given by $E_n = n^2h^2/(8mL^2)$, so we can solve for L to obtain

$$\begin{aligned} E_n &= \frac{n^2h^2}{8mL^2} = \frac{n^2(hc)^2}{8mc^2L^2} \\ L^2 &= \frac{n^2(hc)^2}{8mc^2E_n} \\ L &= \frac{n(hc)}{\sqrt{8mc^2E_n}} = \frac{1\cdot 1240\text{ eV}\cdot\text{nm}}{\sqrt{8\cdot(511\times10^3\text{ eV})\cdot(2.0\text{ eV})}} = 0.43\text{ nm} \end{aligned}$$

Because $E_n \propto n^2$, to go from the $n = 1$ state at 2.0 eV to the $n = 2$ state takes an energy of $(2^2 \cdot 2.0 - 1^2 \cdot 2.0) = 6.0$ eV.

5. You know that the ratio of ^{14}C/^{12}C is 1.3×10^{-12} for "fresh" carbon such as is taken up in plants, animals, *etc.*, and that the half-life of ^{14}C is 5730 years. You are given a 100 g sample of "old" carbon from an archaeological site which has an activity of 400 decays/minute.
A) How old is this sample?
B) If your detector only sees 10% of the decays, and you want to measure the activity with only a 1% error, how long do you need to count for?
Answer: A) The time constant for decay is

$$\lambda = \frac{\log(2)}{t_{1/2}} = \frac{0.693}{5730\ \text{y}} \frac{1\ \text{y}}{365\ \text{d}} \frac{1\ \text{d}}{24\ \text{h}} \frac{1\ \text{h}}{3600\ \text{s}} = 3.83 \times 10^{-12}\ \text{s}^{-1},$$

The intial number of ^{12}C atoms is

$$\frac{100\ \text{g}}{12.0\ \text{g/mol}} \frac{6.02 \times 10^{23}\ \text{atoms}}{1\ \text{mol}} = 5.02 \times 10^{24}\ \text{atoms}$$

The initial number of ^{14}C atoms is $1.3 \times 10^{-12} \cdot 5.02 \times 10^{24} = 6.5 \times 10^{12}$ atoms. The activity of 100 g of "fresh" carbon due to its ^{14}C mass fraction is thus

$$R_0 = N_0 \lambda = (6.5 \times 10^{12}\ \text{atoms}) \cdot (3.83 \times 10^{-12}\ \text{s}^{-1}) = 25\ \text{decays/s}$$

or 1500 decays/minute. The activity of the sample has thus declined according to $R = R_0 e^{-\lambda t}$ so

$$\begin{aligned} R/R_0 &= e^{-\lambda t} \\ \log(R/R_0) &= -\lambda t \\ t &= -\log(R/R_0)/\lambda = -t_{1/2} \log(R/R_0)/\log(2) \\ &= -(5730\ \text{years}) \log(400/1500)/\log(2) = 10,900\ \text{years} \end{aligned}$$

B) If the decay rate of the "old" carbon is 400 decays/minute and you can detect only 10% of the decays, you're seeing only 40 decays/minute. Now for counting N events your error is $\sqrt{N}$, so if you want 1% accuracy you want $\sqrt{N}/N = 0.01$ or $0.01 = 1/\sqrt{N}$ or $N = 100^2 = 10^4$. You'll need to count for $(10^4$ counts$)/(40$ counts/minute$) = 250$ minutes to pin the decay rate down to 1% accuracy.

6. Polonium-210 ($^{210}_{84}$Po) undergoes α decay with a half-life of 138 days. Calculate the energy released per decay, and the activity in Curies of 10 μg of Polonium-210.
Answer:
α (or $^4_2$He) decay of $^{210}_{84}$Po is characterized by $^{210}_{84}\text{Po} \rightarrow ^4_2\text{He} + ^{206}_{82}\text{Pb}$ with an a mass loss of 209.982 857-205.974 449-4.002 603=005 805 u, or an energy release of

$$\Delta m\, c^2 = (0.005\,805\ \text{u}) \cdot (931.494\ \frac{\text{MeV}/c^2}{\text{u}}) \cdot c^2 = 5.41\ \text{MeV}$$

The half-life of 138 days involves a decay constant of

$$\lambda = \frac{\ln 2}{t_{1/2}} = \frac{0.693}{(138\ \text{days}) \cdot (24 \cdot 3600\ \text{sec/day})} = 5.81 \times 10^{-8}\ \text{sec}^{-1},$$

and a quantity of 10 μg involves

$$N = (10 \times 10^{-6}\text{ g}) \cdot (\frac{\text{mole}}{210\text{ g}}) \cdot (6.02 \times 10^{23}\,\frac{\text{atoms}}{\text{mole}}) = 2.87 \times 10^{16}\text{ atoms}$$

The initial activity is therefore

$$A = \lambda N = (5.81 \times 10^{-8}\text{ sec}^{-1}) \cdot (2.87 \times 10^{16}\text{ atoms}) = 1.67 \times 10^{9}\,\frac{\text{decays}}{\text{sec}}$$

or

$$(1.67 \times 10^{9}\,\frac{\text{decays}}{\text{sec}}) \cdot (\frac{1\text{ Ci}}{3.7 \times 10^{10}\text{ decays/sec}}) = 0.0451\text{ Ci}.$$

or 45.1 mCi.

7. True story: whistle-blowing former KGB/FSB agent Alexander Litvinenko was slowly murdered in London in 2006, quite probably by drinking tea secretly laced with 10 μg of Polonium-210. If this were to be distributed uniformly through his body of 100 kg mass, calculate the radiation dose (Gray) and exposure (Sievert) he would have received in one week. *Note: the mean free path of the α decay particles in tissue is less than 0.1 mm, so essentially all the energy of each decay is absorbed in the body. The relative biological effectiveness of α radiation is about 15. Also, if you could not solve problem 6, assume that α decay of ^{210}Po releases 10 MeV, and that the activity of 10 μg is 0.1 Ci.*
Answer:
One week or 7 days is short compared to a half-life of 138 days, so we'll assume that the activity remains constant during the week. If all of the activity is absorbed within the body, the total energy absorbed is

$$(1.67 \times 10^{9}\,\frac{\text{decays}}{\text{sec}}) \cdot (5.41 \times 10^{6}\,\frac{\text{eV}}{\text{decay}}) \cdot (1.602 \times 10^{-19}\,\frac{\text{Joules}}{\text{eV}}) \cdot (7 \cdot 24 \cdot 3600\text{ sec})$$

or 875 Joules. The radiation dose is thus (875 Joules)/(100 kg)=8.75 Gray. The exposure is dose times relative biological effectiveness, or (8.75 Gray)·(15)=131 Sievert. Since the lethal dose to half of those who receive it is about 5 Sievert, poisoning with 10 μg was enough to kill Litvinenko even if Polonium-210 was uniformly distributed in his body (if biochemically concentrated in certain organs, even smaller amounts of Polonium-210 could damage those organs).

8. Show that the wavefunction $\psi(x) = Ce^{-ax^2}$ satisfies (under a specific condition) the Schrödinger equation for a harmonic oscillator potential $U = \frac{1}{2}m\omega^2 x^2$. Find a, and find the energy of the state. *Note: you have to derive these things with an explanation of your derivation; don't just write down the final answer and say "because the equation sheet says so."*
Answer: The second derivative of the wavefunction is

$$\begin{aligned}
\frac{d}{dx}\Big(\frac{d}{dx}(C\exp[-ax^2])\Big) &= \frac{d}{dx}\left(-2ax\,(C\exp[-ax^2])\right) \\
&= -2a\,(C\exp[-ax^2]) + (-2ax)(-2ax)(C\exp[-ax^2]) \\
&= (4a^2x^2 - 2a)(C\exp[-ax^2]) \\
&= (4a^2x^2 - 2a)\,\psi(x).
\end{aligned}$$

Therfore Schrödinger says

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi &= E\psi \\
-\frac{\hbar^2}{2m}\left(4a^2x^2 - 2a\right)\psi(x) + \frac{1}{2}m\omega^2x^2\,\psi(x) &= E\,\psi(x) \\
\frac{a\hbar^2}{m} + \left(\frac{1}{2}m\omega^2 - \frac{2a^2\hbar^2}{m}\right)x^2 &= E
\end{aligned}$$

This must be valid for all x, so we require the term in parenthesis to be zero, or

$$\frac{1}{2}m\omega^2 - \frac{2a^2\hbar^2}{m} = 0 \qquad \text{giving} \qquad a = \frac{m\omega}{2\hbar}$$

which in turn lets us solve for the energy of the state:

$$E = \frac{a\hbar^2}{m} = \frac{\hbar^2}{m}\frac{m\omega}{2\hbar} = \frac{1}{2}\hbar\omega.$$

9. Filling no more than two facing pages in a blue book, describe how it is that discrete states in individual atoms become energy bands in multi-atom solids; state what the Fermi-Dirac distribution has to say about state occupancy; and show the relationship between bands and the Fermi energy in conductors, insulators, and semiconductors.

10. How do nuclear weapons work? What's involved in getting the required materials? Fill no more two facing pages in your blue book with your answer.