Physics 251 exam 2, April 1, 2009. You may use a calculator, and the equation sheet that I have provided.
Order of atoms in the periodic table: H, He, Li, Be, B, C, N, O, F, Ne, Na, Mg. . .

1. In a very hot plasma, a particular atom type is present only in a highly ionized form with just one electron bound per atom. Measurements of the absorption spectrum of the plasma shows wavelengths ranging from 2.53 to 3.38 nm. What type of atom is it?

   *Answer:* The shortest wavelength observed corresponds to the largest photon energy observed, which is for the transition from $n = 1$ to $n \rightarrow \infty$. Since in the Bohr model the energy of an electron's state is $E_n = -E_0 Z^2/n^2$, the energy of the $n \rightarrow \infty$ state is zero so the photon energy is equal to the energy of the $n = 1$ state. Therefore

$$\begin{aligned} E_\lambda = \frac{hc}{\lambda} &= E_0 \frac{Z^2}{1^2} \\ Z &= \sqrt{\frac{hc}{\lambda E_0}} = \sqrt{\frac{1240 \text{ eV} \cdot \text{nm}}{(2.53 \text{ nm}) \cdot (13.60 \text{ eV})}} = 6.0 \end{aligned}$$

   so this is a $Z = 6$ atomic number atom, or a carbon atom.

   By the way, I should have simply said a plasma, rather than a very hot one. Cold temperatures correspond to having all atoms in their ground state, while high temperatures correspond to having many atoms in excited states. Generally speaking one needs high temperatures to strip electrons off of atoms, however. . . The longer wavelength corresponds to a $n = 1$ to $n = 2$ transition with

$$E_2 - E_1 = -E_0 Z^2 (\frac{1}{2^2} - \frac{1}{1^2}) = -(13.60 \text{ eV})(6^2)(\frac{1}{4} - 1) = 367.2 \text{ eV}$$

   or $\lambda = hc/E = 3.38$ nm but of course there could also be $n = 6$ to $n = 7$ transitions, and $n = 9$ to $n = 10$ and so on; these transitions would have even lower energies/longer wavelengths. . . Bad of me, but it still doesn't affect the fact that the shortest wavelength corresponds to the highest energy which corresponds to $n = 1 \rightarrow \infty$.

2. A doubly-ionized lithium ion has its electron undergo a transition from the first excited state to the ground state. What is the mean wavelength of the emitted photon? If the measured spectral distribution has a width of $\Delta\lambda = 0.5$ nm, what is the lifetime $\Delta t$ of the first excited state?

   *Answer:* Lithium has $Z = 3$, so $Li^{2+}$ has just one electron. The energy difference between the first excited ($n = 2$) and ground ($n = 1$) states gives rise to the photon energy:

$$E_{\text{photon}} = E_2 - E_1 = -E_0 Z^2 \left(\frac{1}{2^2} - \frac{1}{1^2}\right) = (13.60 \text{ eV}) \cdot 3^2 \cdot \frac{3}{4} = 91.8 \text{ eV}.$$

   The wavelength associated with this photon energy is $\lambda = hc/E = (1240 \text{ eV}\cdot\text{nm})/(91.8 \text{ eV}) = 13.50$ nm. Now because $E = hc/\lambda$, we have

$$\begin{aligned} dE &= d\left(\frac{hc}{\lambda}\right) = -\frac{hc}{\lambda^2}\, d\lambda = -E\, \frac{d\lambda}{\lambda} \\ \frac{dE}{E} &= -\frac{d\lambda}{\lambda} \end{aligned}$$

so we have

$$\Delta E = E\,\frac{\Delta\lambda}{\lambda} = (91.8\text{ eV})\cdot\left(\frac{0.5\text{ nm}}{13.50\text{ nm}}\right) = 3.4\text{ eV}$$

where we can ignore the minus sign because it just tells us that longer-than-mean wavelengths correspond to lower-than-mean photon energies, but the magnitude of the deviation from the mean is unsigned. Finally, the uncertainty principle gives $(\Delta E)(\Delta t) \geq \hbar/2$, or

$$\Delta t = \frac{\hbar}{2\,\Delta E} = \frac{h}{4\pi\,\Delta E} = \frac{4.14\times 10^{-15}\text{ eV}\cdot\text{sec}}{4\pi\,(3.4\text{ eV})} = 9.7\times 10^{-17}\text{ seconds.}$$

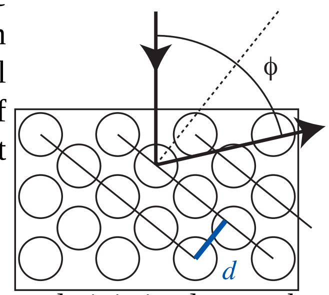
3. An electron with a kinetic energy of 5.0 eV travels along before encountering a barrier with a height of 8.0 eV. What's the de Broglie wavelength of the electron before it encounters the barrier? What's a characteristic tunneling distance of the electron inside the barrier?
*Answer:* Before the barrier, we have $\lambda = h/p$ and $E_k = p^2/(2m)$ giving

$$\lambda = \frac{h}{\sqrt{2mE_k}} = \frac{hc}{\sqrt{2mc^2E_k}} = \frac{1240\text{ eV}\cdot\text{nm}}{\sqrt{2\cdot(511\times 10^3\text{ eV})\cdot(5.0\text{ eV})}} = 0.58\text{ nm}$$

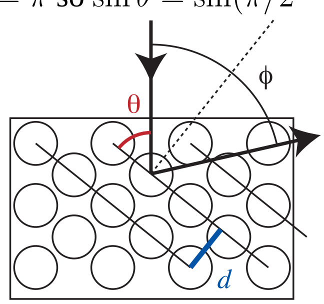
for the de Broglie wavelength. When $E < U$, the wave function falls off as $\psi = \psi_0\exp[-\alpha x]$ with $\alpha = \sqrt{2m(U-E)}/\hbar$, so a characteristic distance is $\delta = 1/\alpha$ or

$$\delta = \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U-E)}} = \frac{hc}{2\pi\sqrt{2mc^2(U-E)}} = \frac{1240\text{ eV}\cdot\text{nm}}{2\pi\sqrt{2\cdot(511\times 10^3\text{ eV})\cdot(8.0-5.0\text{ eV})}} = 0.11\text{ nm}$$

4. An electron beam with kinetic energy $K$ is at normal incidence to a crystal. The first diffraction peak appears at an angle $\phi$ from the incident beam direction, due to a periodicity $d$ in the crystal (a periodicity of $d$ means that atomic planes have a spacing of $d$). Find an analytical expression for $\phi$. Find a numerical result for $\phi$ with $K = 10$ eV and $d = 0.2$ nm.




*Answer:* With a kinetic energy of only 10 eV, this is a very non-relativistic electron beam. From Bragg's law we know $2d\sin\theta = \lambda$ for the $n = 1$ diffraction order, and from the following diagram we see that $2\theta + \phi = \pi$ so $\sin\theta = \sin(\pi/2 - \phi/2) = \cos(\phi/2)$.



In addition, we have $K = p^2/(2m)$ or $p = \sqrt{2mK}$ and the de Broglie wavelength $\lambda = h/p$. If we put these things together we have

$$\begin{aligned} 2d\sin\theta &= \lambda \\ 2d\cos(\frac{\phi}{2}) &= \frac{h}{\sqrt{2mK}} \\ \phi &= 2\arccos\left(\frac{h}{2d\sqrt{2mK}}\right) = 2\arccos\left(\frac{hc}{2d\sqrt{2mc^2K}}\right) \\ &= 2\arccos\left(\frac{1240 \text{ eV}\cdot\text{nm}}{2(0.2 \text{ nm})\sqrt{2\cdot(511\times10^3 \text{ eV})\cdot(10 \text{ eV})}}\right) = 28.3^\circ \end{aligned}$$

5. Solve for me the problem of the ground state of an infinite quantum well of width $L$ (that is, $U = 0$ within $0 \le x \le L$, and $U \to \infty$ otherwise) from first principles. While the equation sheet happens to show the answer, you must *explain* all the steps of applying the Schrödinger equation to this problem, and *derive/justify* for me the wave function, its normalization, and the energy of the ground state.
*Answer:* As discussed in class. Important elements that should be in your derivation:

   (a) Why is $\psi = 0$ outside the quantum well, and also at the edges of $0 \le x \le L$? Roughly speaking, because when $U \to \infty$ we must have $\psi \to 0$ if we are to avoid infinite energy solutions in the $U\psi = E\psi$ part of the Schrödinger equation.

   (b) Why $\psi = A\sin(\pi x/L)$ is a good solution. We need a function that gives $\psi = 0$ at the walls ($x = 0$ and $x = L$), and reproduces a free particle with $\psi = A\sin(kx) + B\cos(kx)$ inside. A half-sine-wave with $\psi = A\sin(\pi x/L)$ does this with the "least wiggles" (that is, longest wavelength or lowest energy).

   (c) Plugging it in the Schrödinger equation to get the energy. We'll need the second derivative of $\psi$:
   $$\frac{d^2}{dx^2}A\sin(\pi\frac{x}{L}) = -A(\frac{\pi}{L})^2\sin(\pi\frac{x}{L}) = (-\frac{\pi^2}{L^2})\psi.$$
   Putting this in the Schrödinger equation gives
   $$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi &= E\psi \\ -\frac{\hbar^2}{2m}(-\frac{\pi^2}{L^2})\psi + 0\psi &= E\psi \\ E &= \frac{\pi^2\hbar^2}{2mL^2} \end{aligned}$$

   (d) Doing the integral of $\int_{-\infty}^{+\infty}|\psi|^2dx = 1$ to find that $A = \sqrt{2/L}$. Done in lecture notes, and you have the solution to the integral of $\int\sin^2(ax)\,dx$ on your equation sheet.