

1. Calculate the de Broglie wavelength for an electron with a kinetic energy of 20 eV and 200 keV.

Answer: The 20 eV electron is non-relativistic, so we have $E_k = p^2/2m$ and $p = \sqrt{2mE_k}$ giving a wavelength of

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}} = \frac{hc}{\sqrt{2mc^2E_k}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2 \cdot (511 \times 10^3 \text{ eV}) \cdot (20 \text{ eV})}} = 0.274 \text{ nm}.$$

The 200 keV electron energy is close enough to the rest mass that we must use a relativistic approach. We know $E_k = (\gamma - 1)mc^2$ so

$$\gamma = 1 + \frac{E_k}{mc^2} = 1 + \frac{200 \text{ keV}}{511 \text{ keV}} = 1.39$$

and then from $\gamma \equiv 1/\sqrt{1-\beta^2}$ we can find

$$\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 1/(1.39^2)} = 0.695$$

We can then use $\lambda = h/p$ and $p = \gamma mv = \gamma\beta mc$ to find

$$\lambda = \frac{h}{\gamma\beta mc} = \frac{hc}{\gamma\beta mc^2} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.39 \cdot 0.695 \cdot (511 \times 10^3 \text{ eV})} = 0.0025 \text{ nm}.$$

2. Consider an atom with a nucleus consisting of 2 protons, and instead of an electron a muon (207 times as heavy as an electron, or $m_\mu = 207m_e$). Calculate the energy of the ground state and the first excited state, and the wavelength of light associated with the transition.

Answer: The reduced mass is

$$m_r = \frac{2m_p \cdot m_\mu}{2m_p + m_\mu} = \frac{2m_p \cdot 207m_e}{2m_p + 207m_e} = \frac{2 \cdot (939 \times 10^3 \text{ keV}) \cdot 207 \cdot (511 \text{ keV})}{2 \cdot (939 \times 10^3 \text{ keV}) + 207 \cdot (511 \text{ keV})} = 1.00 \times 10^5 \text{ keV}.$$

Since $E_0 = me^4/8\epsilon_0^2h^2$ scales with m , we can scale E_0 with m_e to the reduced mass by multiplying by m_r/m_e . The Bohr energy then becomes

$$E_n = -\frac{Z^2}{n^2} E_0 \frac{m_r}{m_e} = -\frac{1}{n^2} 2^2 (13.60 \text{ eV}) \frac{1.00 \times 10^5 \text{ keV}}{511 \text{ keV}} = -\frac{1}{n^2} 1.065 \times 10^4 \text{ eV}$$

so that the energy of the $n = 1$ state is 10,650 eV while the energy of the $n = 2$ state is 2,660 eV. The energy difference is 7,990 eV giving a photon wavelength of

$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV} \cdot \text{nm}}{7990 \text{ eV}} = 0.155 \text{ nm}.$$

3. A beam of atoms with mass $m = 1 \times 10^{25}$ kg and velocity of 1000 m/s is directed from the right, straight into a laser beam coming from the left; the laser is tuned to a wavelength just above the $\lambda = 500$ nm absorption resonance of the atom at rest. Why will still-moving atoms absorb a photon while stationary atoms will not? When the atoms absorb a photon, they remain excited for an average of 1×10^8 seconds before they undergo spontaneous emission and return to their ground state. Calculate the distance it takes to bring an atom to a stop using such a system. Make sure your solution includes some brief explanations rather than just equations. *Typo! It was supposed to have been 10^{-25} kg and 10^{-8} seconds! Corrected by the T.A.s during the exam but of course if students used the silly, incorrect values I will not penalize them for my typo. . .*

Answer: This is Serway problem 4.43. When the atom is moving towards the laser, the laser light will appear blue-shifted through the Doppler effect; as a result, only those atoms moving towards the laser absorb photons. When the atom absorbs a photon, it receives a momentum kick $\Delta p = E/c$ where $E = hc/(500 \text{ nm})$ is the photon energy and $p = E/c$ gives the momentum of a photon. The excited atom will emit a photon $\Delta t = 1 \times 10^{-8}$ seconds later, and those photons will come out in random directions so the *average* momentum kick over many spontaneous emissions is zero (there will be as many momentum kicks to the left as to the right, for example). The atom cannot absorb another photon at the same energy when it is excited, so the timescale before we can have another momentum kick is Δt . This gives a force of

$$F = ma = \frac{\Delta P}{\Delta t} = \frac{E/c}{\Delta t} = \frac{h}{\lambda \Delta t}$$

or a deceleration of (using all SI units)

$$a = -\frac{h}{m\lambda \Delta t} = -\frac{6.63 \times 10^{-34}}{(1 \times 10^{-25}) \cdot (500 \times 10^{-9}) \cdot (1 \times 10^{-8})} = -1.3 \times 10^6 \text{ m/s}^2.$$

Now that we know the acceleration, we can figure out that the atoms slow down over a distance of

$$\begin{aligned} v_f^2 &= v_0^2 + 2a(x - x_0) \\ (x - x_0) &= \frac{v_f^2 - v_0^2}{2a} = \frac{0^2 - (10^3 \text{ m/s})^2}{2(-1.3 \times 10^6)} = 0.38 \text{ m} \end{aligned}$$

4. An electron is in a finite square well of width $L = 0.6$ nm and $U = 20.0$ eV. Calculate the characteristic tunneling distance δ , and calculate the energy of the ground state both in the case of $U \rightarrow \infty$ and an approximate value with $U = 20.0$ eV.

Answer: For the infinite square well, we have

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2} = n^2 \frac{(hc)^2}{8mc^2L^2} = n^2 \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8 \cdot (511 \times 10^3 \text{ eV}) \cdot (0.6 \text{ nm})^2} = n^2(1.05 \text{ eV})$$

so of course $E_1 = 1.05$ eV. For $U = 20.0$ eV, we have

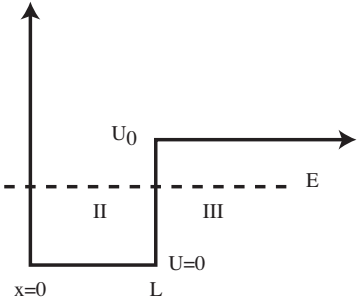
$$\begin{aligned} \delta &= \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U - E)}} = \frac{hc}{2\pi\sqrt{2mc^2(U - E)}} \\ &= \frac{1240 \text{ eV} \cdot \text{nm}}{2\pi\sqrt{2 \cdot (511 \times 10^3 \text{ eV}) \cdot (20.0 - 1.05 \text{ eV})}} = 0.045 \text{ nm} \end{aligned}$$

The approximate result for the energy in the finite quantum well is then

$$E_n \simeq n^2 \frac{(hc)^2}{8mc^2(L + 2\delta)^2} = n^2 \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8 \cdot (511 \times 10^3 \text{ eV}) \cdot (0.5 + 2 \cdot 0.045 \text{ nm})^2} = n^2(0.78 \text{ eV})$$

so the ground state energy is reduced from 1.05 eV to 0.78 eV.

- Consider the potential shown at right. Describe the functional form of wavefunctions for the three regions ψ_I , ψ_{II} , and ψ_{III} . Start from the Schrödinger equation to justify your choice of wavefunction forms in each region, and to derive the coefficient that multiplies distance x in each region (this should be a function of E and U , among other things). Show that there exist solutions at discrete energies. Don't write a wordy, multipage essay; but you should have some short sentences of explanation at various steps.



Answer: Because the potential goes to ∞ at $x \leq 0$, we need to have $\psi_I = 0$ in order not to have a blowup of $U\psi$ leading to a blowup of E in $E\psi$ in the Schrödinger equation. In region II we have to have $\psi_{II}(x = 0) = 0$; beyond that, we have $U = 0$ so the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + U\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \quad \text{with} \quad k = \frac{\sqrt{2mE}}{\hbar}.$$

Now $\psi = A \sin(kx)$ and $\psi = A \cos(kx)$ both have the property of having second derivatives be $-k^2\psi$ and thus satisfying the above differential equation, but only the sine solutions have $\psi(x = 0) = 0$ as required where $U(x \rightarrow 0) \rightarrow \infty$. Therefore $\psi_{II} = A \sin(kx)$. In region III ($x > L$) the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + U\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{2m(U - E)}{\hbar^2}\psi = \alpha^2\psi \quad \text{with} \quad \alpha = \frac{\sqrt{2m(U - E)}}{\hbar}.$$

Wavefunctions of $\psi = Be^{-\alpha x}$ and $\psi = Be^{+\alpha x}$ both have the property of having second derivatives of $+\alpha^2\psi$, but only $\psi_{III} = Be^{-\alpha(x-L)}$ has the property of decreasing in the classically forbidden region while $e^{+\alpha(x-L)}$ would increase unbounded and again lead to infinite energy solutions. Again, we have $\psi_{II} = A \sin(kx)$ with $k = \sqrt{2mE}/\hbar$ and $\psi_{III} = Be^{-\alpha(x-L)}$ with $\alpha = \sqrt{2m(U - E)}/\hbar$. Now we must satisfy boundary conditions to make the wave smooth:

$$\psi_{II}(x = L) = \psi_{III}(x = L) \quad \text{and} \quad \frac{d\psi_{II}}{dx}\bigg|_{x=L} = \frac{d\psi_{III}}{dx}\bigg|_{x=L}$$

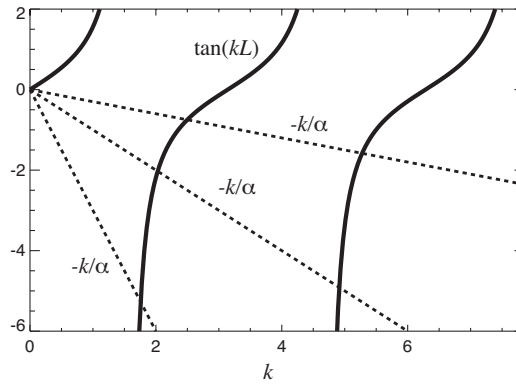
or

$$A \sin(kL) = Be^{-\alpha \cdot 0} = B \quad \text{and} \quad Ak \cos(kL) = -\alpha Be^{-\alpha \cdot 0} = -\alpha B$$

and if we divide these two equations we have

$$\frac{A \sin(kl)}{Ak \cos(kl)} = \frac{B}{-\alpha B} \quad \Rightarrow \quad \tan(kL) = -\frac{k}{\alpha}.$$

Viewing this as a graphical solution of how $\tan(kL)$ scales with k , versus how $-k/\alpha$ scales with k , gives



As you can see, no matter what slope $-1/\alpha$ we pick, we're going to have only discrete points where $\tan(kL)$ intersects with $-k/\alpha$, and thus discrete energy solutions.