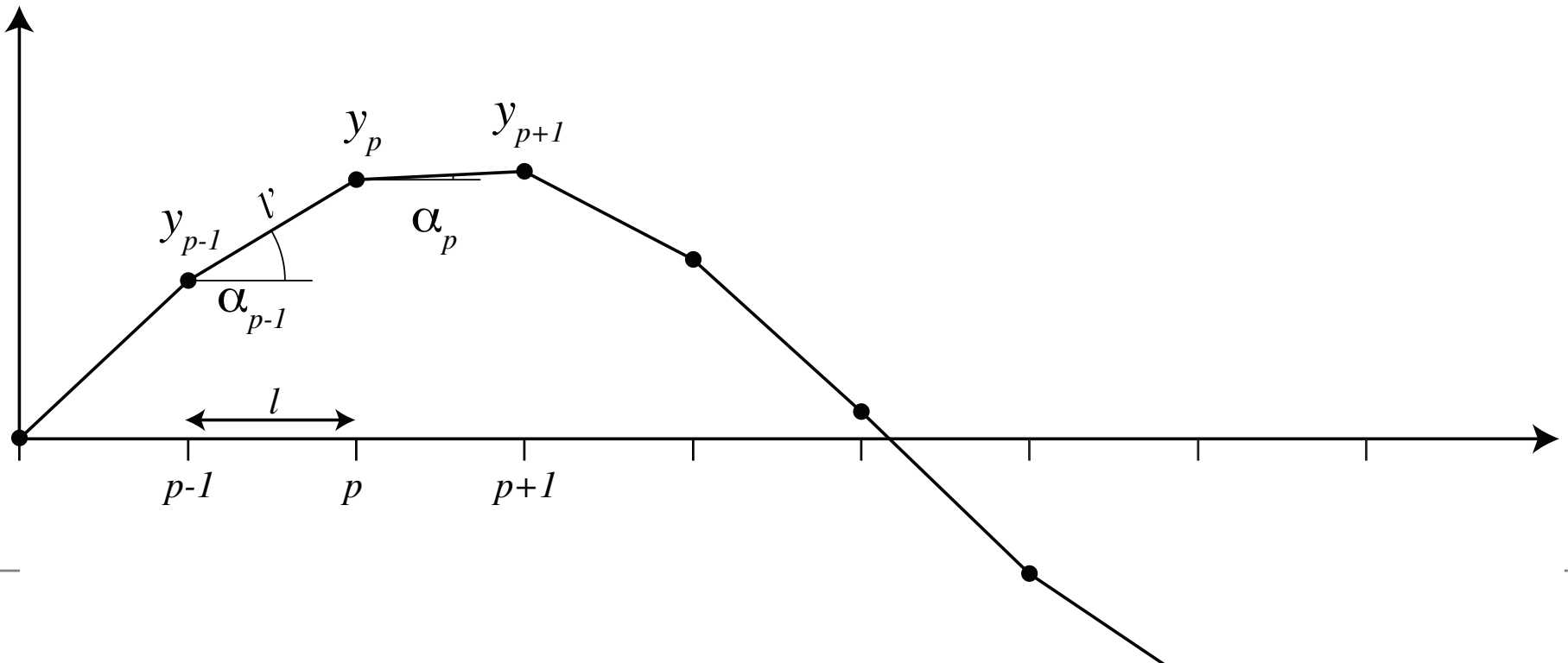


Coupled oscillators: from 2 to oodles

Let's move from two coupled pendulums to N coupled oscillators.

- Now each oscillator experiences a force from a neighbor on each side. We'll assume that the coupling between oscillators is dominant, and talk about them being on a string with tension T .
- We'll index each oscillator with an integer p for position.
- Each position will be a distance ℓ apart along the axis, or ℓ' along the string.



Oodles I

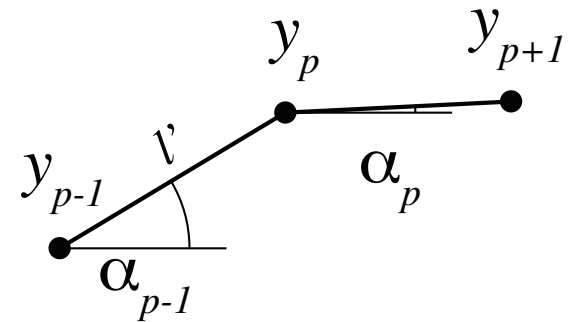
Angles between points:

$$(1) \quad \alpha_{p-1} = \tan^{-1} \left(\frac{y_p - y_{p-1}}{\ell} \right) \simeq \frac{y_p - y_{p-1}}{\ell}$$

What's ℓ' ? we can write

$$(2) \quad \ell' = \frac{\ell}{\cos \alpha} \simeq \frac{\ell}{1 - \alpha^2/2} \simeq \ell(1 + \alpha^2/2)$$

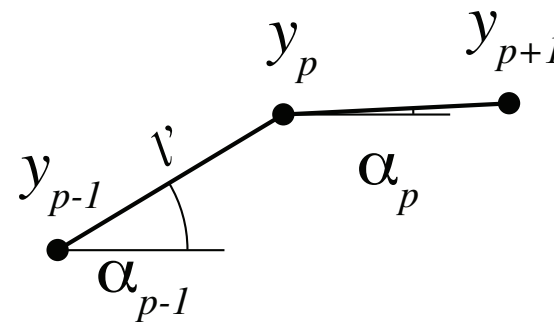
So $\ell' - \ell = \ell\alpha^2/2$. We'll ignore α^2 effects throughout. If $\ell' \simeq \ell$, then the tension T on each point is the same.



Oodles II

- Since the tension is independent of α (in the limit $\alpha^2 \ll 1$), we say T is a constant.
- Net force in x on p :

$$\begin{aligned} F_x &= -T \cos \alpha_{p-1} + T \cos \alpha_p \\ &\simeq T \left(-1 + \frac{\alpha_{p-1}^2}{2} + 1 - \frac{\alpha_p^2}{2} \right) \\ (3) \quad &\simeq \frac{T}{2} (\alpha_{p-1}^2 - \alpha_p^2) \end{aligned}$$

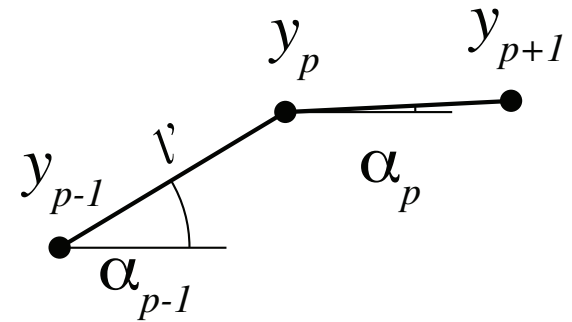


- Since this depends on α^2 , we will ignore this force; each point stays at a constant position x_p .

Oodles III

Consider now the net force in y :

$$\begin{aligned} F_y &= -T \sin \alpha_{p-1} + T \sin \alpha_p \\ &\simeq T(\alpha_p - \alpha_{p-1}) \\ &\simeq \frac{T}{\ell}(y_{p+1} - y_p - y_p + y_{p-1}) \\ (4) \quad &\simeq \frac{T}{\ell}(-2y_p + y_{p+1} + y_{p-1}) \end{aligned}$$



where we have made use of the result of Eq. 1 of

$$\alpha_{p-1} \simeq \frac{y_p - y_{p-1}}{\ell}$$

Oodles IV

We have found that we can ignore F_x , so that x_p is a constant. In the y direction, the net force of Eq. 4 produces acceleration:

$$m \frac{d^2 y_p}{dt^2} = \frac{T}{\ell} (-2y_p + y_{p+1} + y_{p-1})$$

$$(5) \quad \frac{m\ell}{T} \frac{d^2 y_p}{dt^2} + 2y_p - (y_{p+1} + y_{p-1}) = 0$$

We have a puzzle here: we have a differential equation in y_p , but it is also coupled to neighboring positions y_{p-1} and y_{p+1} . Still, this looks enough like a harmonic oscillator that we will assume that we are looking for solutions of the form $y_p = A_p e^{i\omega t}$, in which case Eq. 5 becomes

$$(6) \quad -\omega^2 \frac{m\ell}{T} A_p e^{i\omega t} + 2A_p e^{i\omega t} + (A_{p+1} + A_{p-1}) e^{i\omega t} = 0$$

We see that ω^2 has the same dimensions as $T/(m\ell)$, so we will make the definition

$$(7) \quad \omega_0^2 \equiv \frac{T}{m\ell}$$

Oodles V

With the definition of Eq. 7 of $\omega_0^2 \equiv T/(m\ell)$, we can rewrite Eq. 5 as

$$(8) \quad \frac{d^2 y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

If we return to our assumption of $y_p = A_p e^{i\omega t}$, this becomes

$$-\omega^2 A_p e^{i\omega t} + 2\omega_0^2 A_p e^{i\omega t} - \omega_0^2 (A_{p+1} + A_{p-1}) e^{i\omega t} = 0$$

$$\text{or} \quad (-\omega^2 + 2\omega_0^2) A_p - \omega_0^2 (A_{p+1} + A_{p-1}) = 0$$

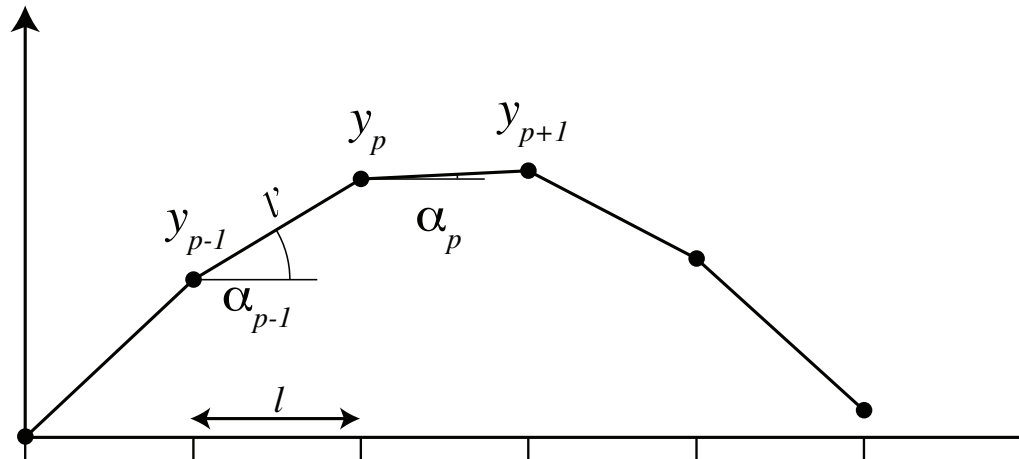
$$(9) \quad \text{or} \quad (-\omega^2 + 2\omega_0^2) A_p = \omega_0^2 (A_{p+1} + A_{p-1})$$

This gives us the relationship

$$(10) \quad \frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

Oodles VI

Let's think again about the relationship between amplitudes of successive points:



Let's test a reasonable guess for the relationship between amplitudes of successive points:

$A_p = C \sin(p\theta)$. That is, θ is an increment of amplitude from one point to another for a standing wave solution, as we'll see later. If we consider a standing wave with fixed ends such that $A_{p=0} = 0$ and $A_{p=N+1} = 0$, we can say that $(N + 1)\theta = n\pi$ with $(n = 1, 2, 3, \dots)$ and thus write the amplitude A_p as

$$(11) \quad A_p = C_n \sin\left(\frac{pn\pi}{N+1}\right) \quad \text{with } n = 1, 2, 3, \dots$$

Oodles VII

With the assumption $A_p = C \sin(p\theta)$, we have

$$\begin{aligned}
 A_{p-1} + A_{p+1} &= C \left[\sin((p-1)\theta) + \sin((p+1)\theta) \right] \\
 &= C \left[\sin(p\theta) \cos(-\theta) + \cos(p\theta) \sin(-\theta) + \sin(p\theta) \cos(\theta) + \cos(p\theta) \sin(\theta) \right] \\
 &= C \left[\sin(p\theta) \cos(\theta) - \cos(p\theta) \sin(\theta) + \sin(p\theta) \cos(\theta) + \cos(p\theta) \sin(\theta) \right] \\
 &= 2C \sin(p\theta) \cos(\theta) \\
 (12) \qquad &= 2A_p \cos \theta
 \end{aligned}$$

where we have used the trig identity for $\sin(\alpha + \beta)$, and $\cos(-\theta) = \cos(\theta)$, and $\sin(-\theta) = -\sin(\theta)$ to reproduce French Eq. 5-21. If we now use the result $(N + 1)\theta = n\pi$ arrived at before Eq. 11, and Eq. 10, we have

$$(13) \qquad \frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} = 2 \cos\left(\frac{n\pi}{N+1}\right)$$

Oodles VIII

Again, we had from Eq. 13 the result

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} = 2 \cos\left(\frac{n\pi}{N+1}\right)$$

Let's define $\beta \equiv n\pi/(N+1)$ and solve for the variable frequency ω :

$$\begin{aligned} -\omega^2 + 2\omega_0^2 &= 2\omega_0^2 \cos \beta \\ \omega^2 &= 2\omega_0^2(1 - \cos \beta) \\ &= 4\omega_0^2 \frac{1 - \cos \beta}{2} \\ &= 4\omega_0^2 \sin^2\left(\frac{\beta}{2}\right) \\ (14) \quad \omega &= 2\omega_0 \sin\left(\frac{\beta}{2}\right) = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \end{aligned}$$

were we have made use of the trig identity $\sin^2 \beta/2 = (1/2)(1 - \cos \beta)$ in arriving at the final result.

Oodles IX

What hath we wrought? We have from Eq. 11

$$y_p = A_p e^{i\omega t} = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega t}$$

and from Eq. 14 the result

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) = 2\omega_0 \sin\left(\frac{n}{N+1} \frac{\pi}{2}\right) \quad \text{with } n = 1, 2, 3, \dots$$

so we should really write the position of the p^{th} particle as $y_{pn}(t)$. Now let us consider the frequencies allowed by Eq. 14. If we increase n from 1 up to $N+1$ (the number of oscillating points, because p goes from 0 to $N+1$), we will have unique values of ω_n . However, when n goes to $N+2$, we have

$$\frac{N+2}{N+1} = \frac{N+1}{N+1} + \frac{1}{N+1}$$

but since $\sin(\pi/2 + \beta) = \sin(\pi/2 - \beta)$, we'll have the same result for Eq. 14 for $n = (N+1) + 1$ as for $n = (N+1) - 1$. That is, we have only $n = 1, 2, \dots, N+1$ unique frequencies.

Oodles X

We have determined that we have only $n = 1, 2, \dots, N + 1$ unique frequencies in the result of Eq. 11 of

$$y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$$

In fact, when $n = N + 1$, the amplitude is $C_{N+1} \sin(p\pi)$, and since p is an integer the amplitude for $n = N + 1$ is zero. (Well, duh; this was built into our assumption that $(N + 1)\theta = n\pi$ when arriving at Eq. 11). So really we have only $n = 1, 2, \dots, N$ unique frequencies with non-zero amplitude. Also, just as we found that there are only N unique frequencies, the same argument applied to the amplitudes again shows that there are only N unique results. We've learned something important:

N oscillators between fixed points have N allowed modes of oscillation

and we should write Eq. 11 as

$$(15) \quad y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t} \quad \text{with } n = 1, 2, \dots, N$$

Standing waves

Again, we have Eq. 15 of

$$y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t} \quad \text{with } n = 1, 2, \dots, N \text{ and } p = 0, 1, \dots, N+1$$

Let's consider the $n = 1$ case:

$$y_{p1} = C_1 \sin\left(\frac{p}{N+1}\pi\right) e^{i\omega_1 t}$$

Each point p oscillates at the frequency ω_1 with an amplitude of C_1 times $\sin \theta$ with θ going from 0 to π . This is a standing wave with maximum amplitude in the center (French Fig. 5-13).

The $n = 2$ case looks like

$$y_{p2} = C_2 \sin\left(\frac{p}{N+1}2\pi\right) e^{i\omega_2 t}$$

which goes like $\sin \theta$ with $\theta = 0 \rightarrow 2\pi$. We have a node in the middle (French Fig. 5-14). Get the pattern?

Strings to springs

Let's now consider $p = 1, 2, \dots, N$ masses m coupled by springs with spring constant $k = m\omega_0^2$ (points $p = 0$ and $p = N + 1$ will be fixed points at either end of the system). Let y_p represent the displacement of each point from its equilibrium position. The force that a point p feels is given by the relative spring force it feels from each side:

$$m \frac{d^2 y_p}{dt^2} = k(y_{p+1} - y_p) - k(y_p - y_{p-1})$$

$$\frac{d^2 y_p}{dt^2} = \frac{k}{m}(-2y_p + y_{p+1} + y_{p-1})$$

$$\frac{d^2 y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0$$

This is exactly the same mathematical form as we had in Eq. 8! In that case we had $\omega_0^2 = T/(m\ell)$, while now we have $\omega_0^2 = k/m$; and we interpreted y_p as the vertical displacement of a string stretched horizontally rather than the displacement from a longitudinal equilibrium position, but everything we've done above also applies to a series of N masses coupled by springs.

Oodles of oodles

Let's consider the case of very large values of N . For standing waves on a string, the total length of the string is $L = (N + 1)\ell$ and its total mass is $M = Nm$ with a mass per unit length of $\mu \equiv m/\ell$. Now our spectrum of allowed frequencies was given by Eq. 14 as

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

which in the limit $n \ll N$ becomes

$$\begin{aligned} \omega_n &\simeq 2\sqrt{\frac{T}{m\ell}} \frac{n\pi}{2(N+1)} \\ (16) \quad &\simeq \sqrt{\frac{T}{m/\ell}} \frac{n\pi}{\ell(N+1)} = \sqrt{\frac{T}{\mu}} \frac{n\pi}{L}. \end{aligned}$$

We see that heavier strings have lower frequencies for the same length and tension, which tells us about things like how to build guitars and pianos.

Frequency cutoff

The highest mode is with $n = N$, which from Eq. 14 gives in the limit $N \gg 1$

$$\omega_n = 2\omega_0 \sin\left(\frac{N\pi}{2(N+1)}\right) \simeq 2\omega_0 \sin\left(\frac{\pi}{2}\right) \simeq 2\omega_0$$

What does the motion look like at this frequency? We had from Eq. 15 the result of

$$\begin{aligned} y_{pn}(t) &= C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t} &\Rightarrow & y_{pN} = C_N \sin\left(\frac{pN\pi}{N+1}\right) \\ &= C_N \sin\left(p\pi - \frac{p\pi}{N+1}\right) \end{aligned}$$

so the position of each successive point p has an opposite sign.