The driven harmonic oscillator

No, by “driven” we do not mean “determined...” Let’s consider an object with a linear restoring force and an extra force which is driving it in an oscillatory fashion:

$$m\frac{d^2x}{dt^2} = -kx + F_0 e^{i\omega t}$$

$$m\frac{d^2x}{dt^2} + kx - F_0 e^{i\omega t} = 0 \tag{1}$$

While the undriven object might have its own resonance frequency $\omega_0 = \sqrt{k/m}$, we’re exciting its motion at a (possibly) different frequency ω so we’ll seek solutions involving this frequency, or trial solutions of $x = Ce^{i\omega t+\varphi}$. Putting this into Eq. 1, we have

$$(-\omega^2 m + k - \frac{F_0}{C})Ce^{i\omega t+\varphi} = 0. \tag{2}$$

This must be true at all times t, which means the terms in parenthesis must be zero or

$$-\omega^2 m + k - \frac{F_0}{C} = 0 \tag{3}$$

Driven harmonic oscillator II

Again, we required Eq. 3 or

$$-\omega^2 m + k - \frac{F_0}{C} = 0$$

so that motion with a driving force of $Ce^{i\omega t+\varphi}$ has an amplitude

$$C = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}. \quad (4)$$

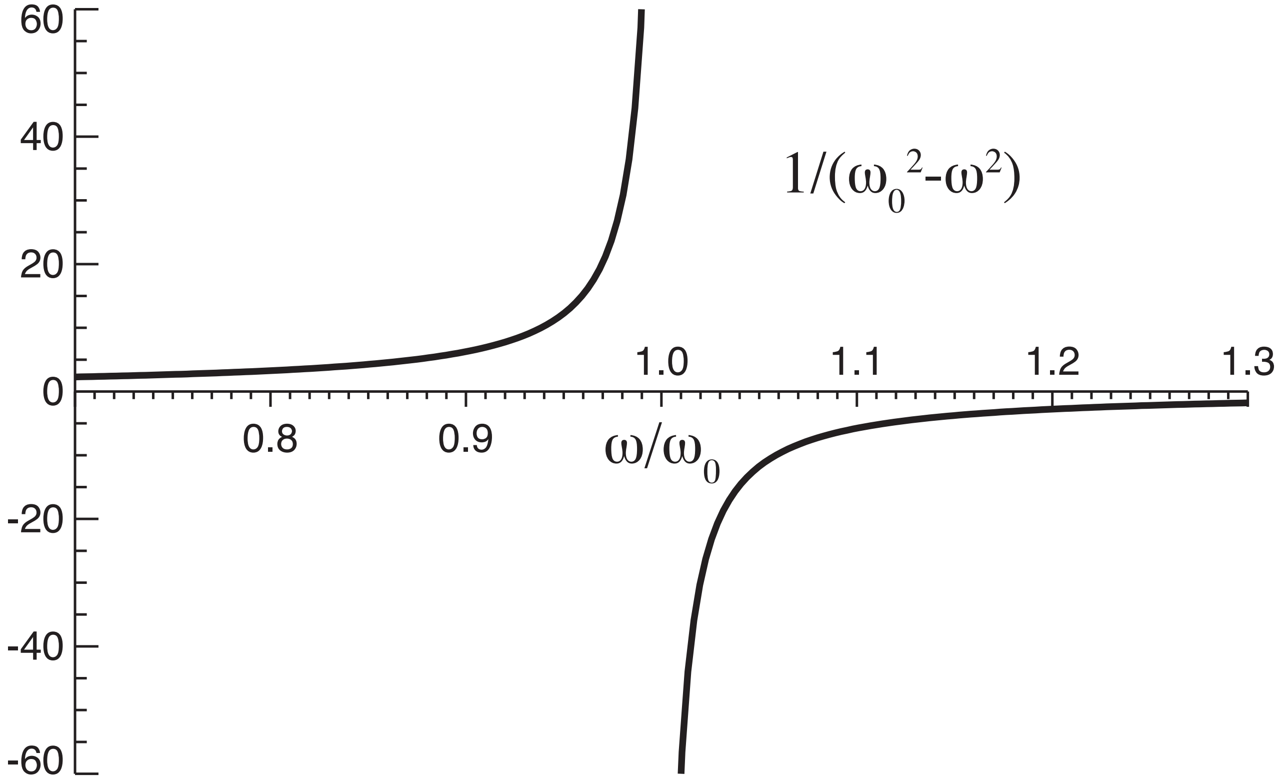
Are we happy with this result?

Driven harmonic oscillator III

Again, we've found that motion with a driving force $F_0 e^{i\omega t+\varphi}$ is given by Eq. 4 as

$$C = \frac{F_0/m}{\omega_0^2 - \omega^2}.$$

What does this function look like?

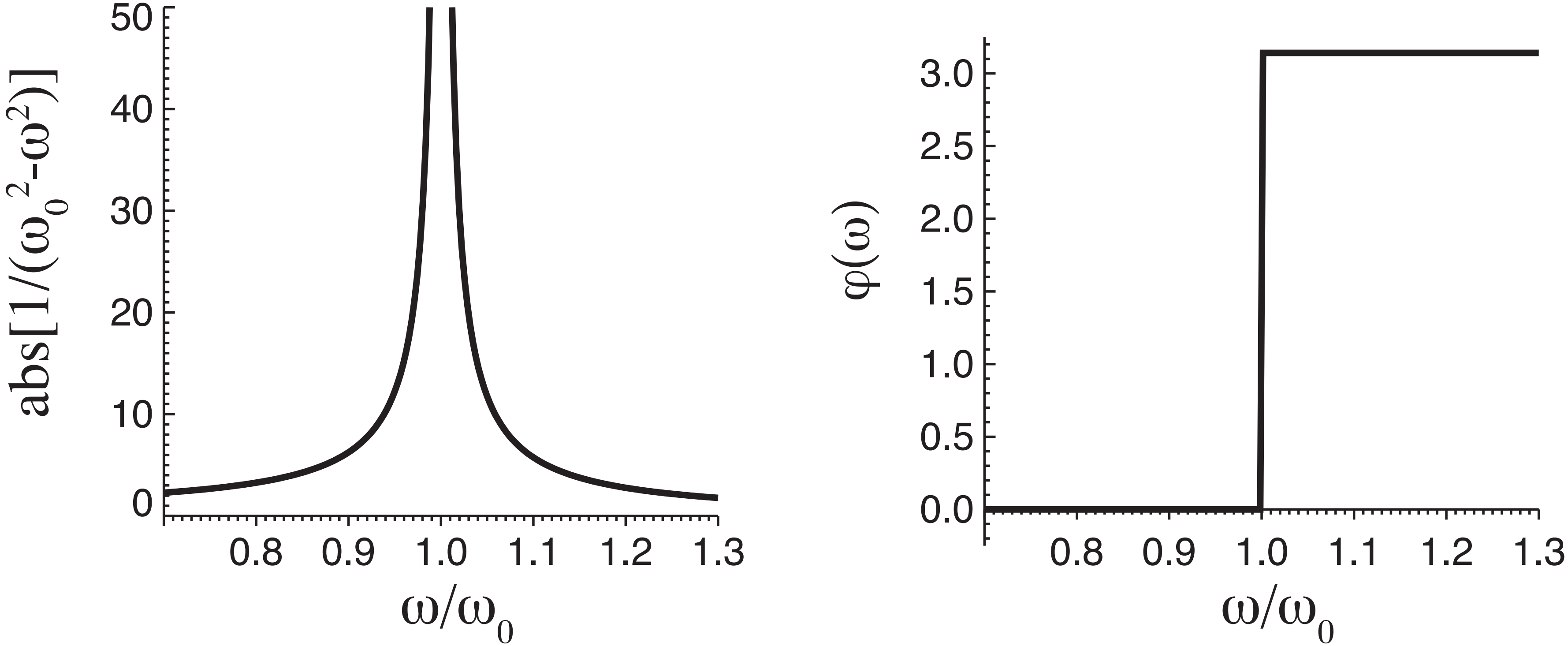


Driven harmonic oscillator IV

We can also describe Eq. 4 as

$$|C| = \frac{F_0/m}{|\omega_0^2 - \omega^2|} \quad \text{with} \quad \varphi(\omega) \tag{5}$$

where $\varphi(\omega < \omega_0) = 0$ and $\varphi(\omega > \omega_0) = \pi$.



Damped, driven harmonic oscillator

When the harmonic oscillator is driven at its resonance frequency, its amplitude tends towards infinity. Do we really believe that this will accurately describe real life? Probably not. A more realistic situation will be to have both a driving force and a damping force, or

$$m\frac{d^2x}{dt^2} = -kx + -b\frac{dx}{dt} + F_0 e^{i\omega t+\varphi}$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}e^{i\omega} \tag{6}$$

where in the second expression we have again assumed $\omega_0^2 \equiv k/m$ and $\gamma \equiv b/m$. Again, we'll assume that the solution is of the form

$$x = \mathrm{Re}[Ae^{i\omega t-\delta}] \tag{7}$$

where we've allowed for a time-independant phase retardation δ in our solution.

Damped, driven harmonic oscillator II

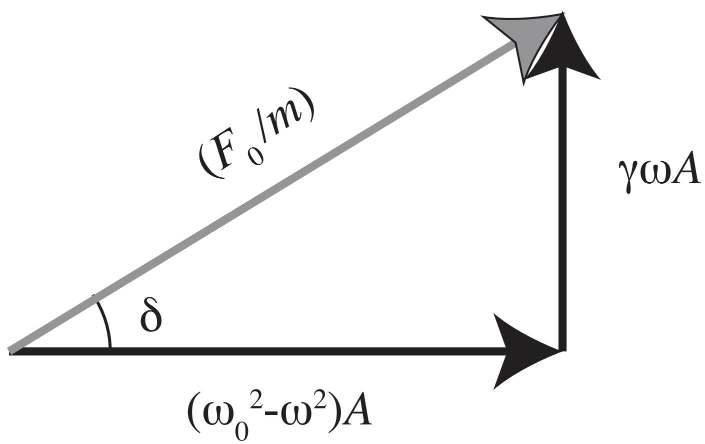
Placing our assumed solution form of Eq. 7 into our differential equation of Eq. 6, we have

(8)
$$(-\omega^2 A + i\gamma\omega A + \omega_0^2 A)e^{i\omega t-\delta} = \frac{F_0}{m}e^{i\omega t}$$

Again, this must be true for any time t, so we require

(9)
$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m}e^{i\delta}$$

This has the same form as $x + iy = Re^{i\theta}$:



Damped, driven harmonic oscillator III

The geometrical interpretation of Eq. 9

$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m} e^{i\delta}$$

implies the following:

$$(\omega_0^2 - \omega^2)A = \frac{F_0}{m} \cos\delta \quad (10)$$

$$\gamma\omega A = \frac{F_0}{m} \sin\delta \quad (11)$$

The ratio of Eq. 11 over Eq. 10 tells us the frequency-dependent phase angle $\delta(\omega)$:

$$\tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2} \quad (12)$$

Damped, driven harmonic oscillator IV

Let's again consider Eq. 9:

$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m}e^{i\delta}$$

If we square both sides by multiplying by their respective complex conjugates, we have

$$(\omega_0^2 - \omega^2)^2 A^2 + (\gamma\omega)^2 A^2 = \left(\frac{F_0}{m}\right)^2$$

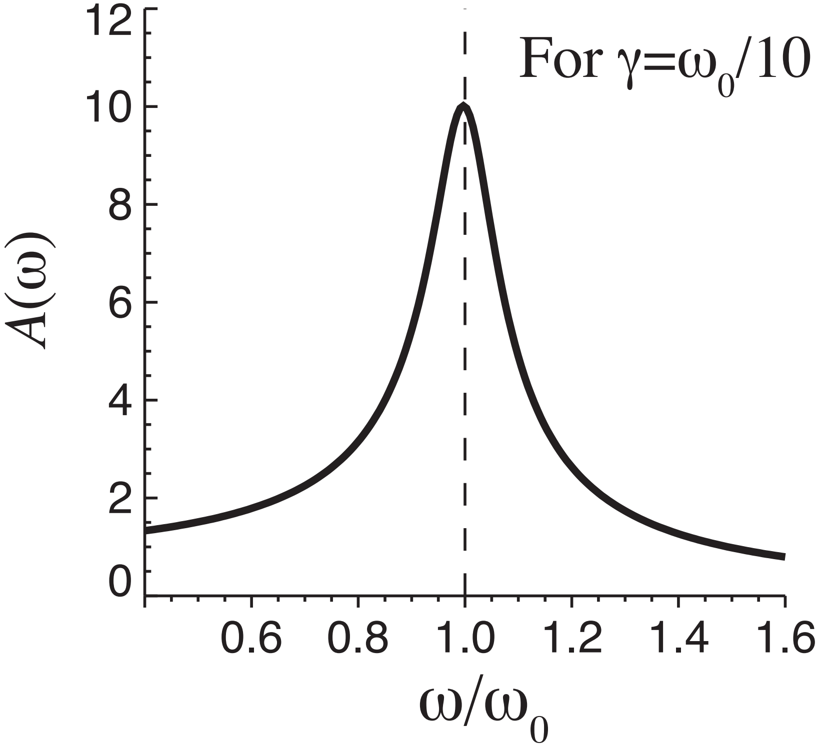
from which we obtain

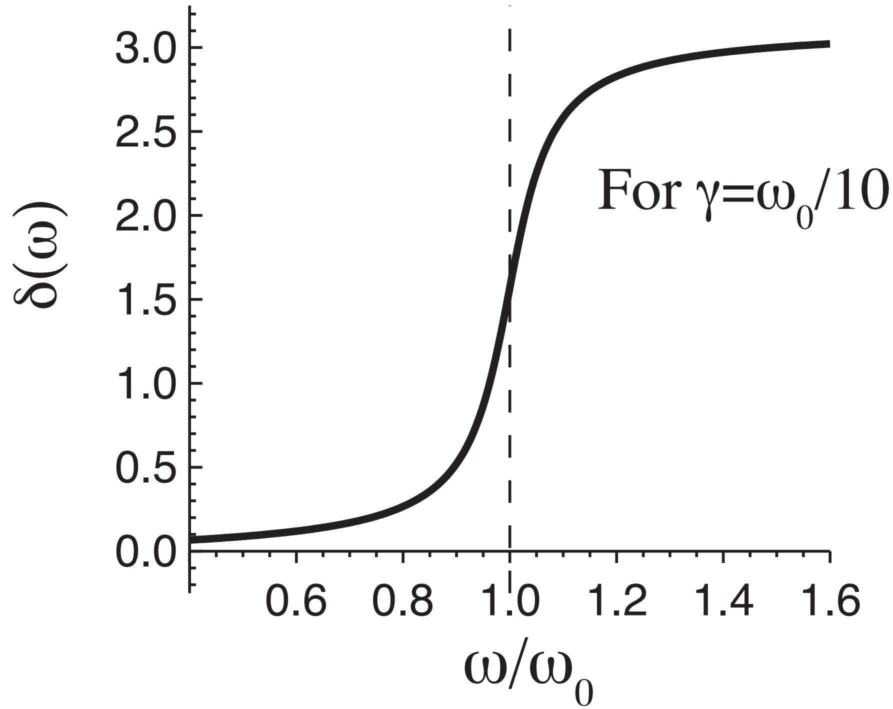
(13)
$$|A(\omega)| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

as the frequency-dependent pure-real representation of the amplitude (with $e^{i\delta(\omega)}$ providing the phase).

Damped, driven harmonic oscillator V

Let's take a look at our solutions for the amplitude and phase of motion for the damped, driven harmonic oscillator:





Note that the motion is in phase with the driving force at low frequencies and 180° out of phase at high frequencies.

Quality factor Q

As with the damped harmonic oscillator, let's characterize our solution in terms of a quality factor $Q \equiv \omega_0/\gamma$ and look again at the solution of Eq. 13 for $|A(\omega)|$:

$$\begin{aligned} |A(\omega)| &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\ &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\omega_0/Q)^2}} \end{aligned} \tag{14}$$

At the point when $\omega = \omega_0$, this reduces to

$$|A(\omega_0)| = \frac{F_0/m}{\sqrt{0 + (\omega_0^2/Q)^2}} = Q\frac{F_0}{m\omega_0^2} = Q\frac{F_0}{k} \tag{15}$$

Thus we see that $Q \equiv \omega_0/\gamma$ factors directly and linearly into the maximum amplitude, so that low damping γ corresponds to large amplitude Q.

Phase of resonant motion

- Let's also examine the phase of the resonant motion from Eq. 12 of

  $$\tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

  As $\omega \to \omega_0$, the denominator goes towards zero so that the argument $\delta(\omega)$ of the tangent function tends towards $\delta(\omega_0) \to \pi/2$.

- When driven at exactly the undamped resonance frequency $\omega_0 = \sqrt{k/m}$, the motion of the object is phase-shifted by 90° relative to the driving force.
- As we noted before, at driving frequencies ω well below the resonant frequency ω_0 the motion is in-phase with the driving force: $\delta(\omega \ll \omega_0) \to 0$.
- At driving frequencies ω well above the resonance frequency ω_0 the motion becomes exactly out-of-phase with the driving force: $\delta(\omega \gg \omega_0) \to \pi$.

Complex amplitude

Let's return to the expression of Eq. 9 of

$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m} e^{i\delta}$$

except we'll now make A be complex ($A \to \tilde{A}$) so that we can throw away the term $e^{i\delta}$ as it is no longer needed for the "bookkeeping" of handling a phase angle. Let's then solve for the complex amplitude $\tilde{A}(\omega)$:

$$\tilde{A}(\omega) = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega} \tag{16}$$

Low frequency limit

Again, we have Eq. 16 of

$$\tilde{A}(\omega) = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

Let's consider the low frequency limit, where $\omega \ll \omega_0$ and $\gamma \ll \omega_0$. In this case we have

$$\begin{aligned} \tilde{A}(\omega \ll \omega_0) &= \frac{F_0/m}{\omega_0^2\left(1 - \frac{\omega^2}{\omega_0^2} + i\frac{\gamma\omega}{\omega_0^2}\right)} \\ &\simeq \frac{F_0}{m\omega_0^2}\left(1 + \frac{\omega^2}{\omega_0^2} - i\frac{\gamma\omega}{\omega_0^2}\right) \end{aligned} \qquad (17)$$

where we have used the binomial expansion (the lowest order Taylor series expansion of $(1+x)^n \simeq 1 + nx$ for $x \ll 1$) to obtain the approximate result. We see again that in the low frequency limit the amplitude is nearly pure real (in-phase motion) and it has a constant term plus a weaker term that increases with ω^2. Later on in the course we will see that this describes the refractive index for visible light, among other things.

High frequency limit

Now let's consider the result of Eq. 16 of

$$\tilde{A}(\omega) = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

in the high frequency limit where $\omega \gg \omega_0$, and the limit of modest damping $\gamma \lesssim \omega_0$. In this case the leading term in the denominator is ω^2 so that the amplitude becomes

(18) $$\tilde{A}(\omega \gg \omega_0) \simeq -\frac{F_0}{m\omega^2}$$

We see two things in the high frequency limit:

1. The amplitude is negative, or since $e^{i\pi} = -1$, it is 180° out of phase with the driving force; and
2. The amplitude decreases as $1/\omega^2$.

As we will see later on in the course, this describes the refractive index for x rays.