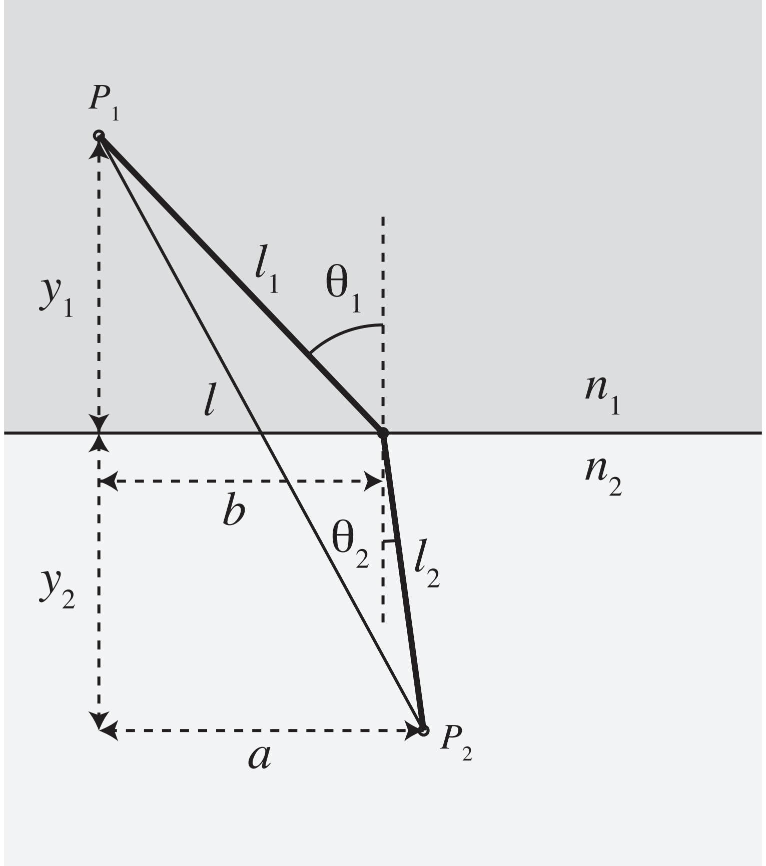
# *Snell's law*

We have shown that light travels according to the principal of stationary phase, or least time. Let's consider how light goes from point $P_1$ to $P_2$ as it travels across a refractive boundary. The time is distance over velocity, or

$$t = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{l_1 n_1}{c} + \frac{l_2 n_2}{c} \tag{1}$$

where we have used the phase velocity $v_p = c/n$. Light will travel along a straight line within one medium, but what about at an interface? What value of $b$ minimizes the travel time/produces a path of stationary phase?



# *Snell's law II*

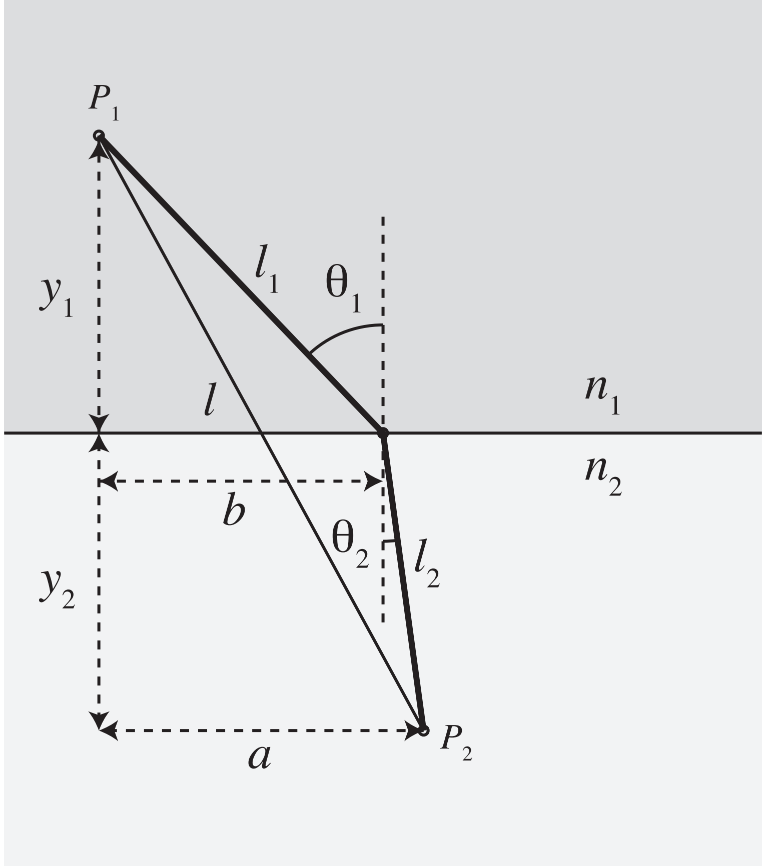
To find the path of least time, let's write the lengths $l_1$ and $l_2$ in terms of $b$:

$$l_1 = \sqrt{y_1^2 + b^2} \qquad l_2 = \sqrt{y_2^2 + (a-b)^2}$$

The travel time of Eq. 1 then becomes

$$(2) \qquad t = \frac{n_1[y_1^2 + b^2]^{1/2}}{c} + \frac{n_2[y_2^2 + (a-b)^2]^{1/2}}{c}$$

To minimize the time, let's set the derivative as we vary $b$ equal to zero: $dt/db = 0$.



# *Snell's law III*

Again, we want to minimize the time of Eq. 2 of

$$t = \frac{n_1[y_1^2 + b^2]^{1/2}}{c} + \frac{n_2[y_2^2 + (a-b)^2]^{1/2}}{c}$$

by setting $dt/db = 0$:

$$\begin{aligned} \frac{dt}{db} = 0 &= \frac{1}{2}\frac{n_1[y_1^2 + b^2]^{-1/2}}{c} 2b + \frac{1}{2}\frac{n_1[y_2^2 + (a-b)^2]^{-1/2}}{c} 2(a-b)(-1) \\ 0 &= n_1 \frac{b}{\sqrt{y_1^2 + b^2}} - n_2 \frac{a-b}{\sqrt{y_2^2 + (a-b)^2}} \qquad (3) \end{aligned}$$

# *Snell's law IV*

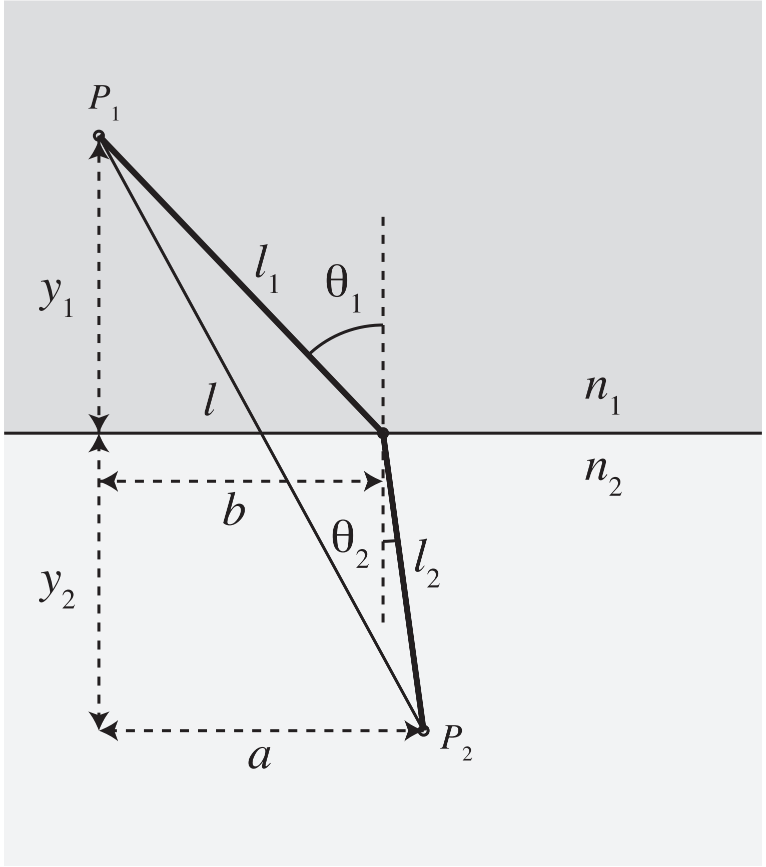
Again our condition for minimizing the time was given in Eq. 3:

$$0 = n_1 \frac{b}{\sqrt{y_1^2 + b^2}} - n_2 \frac{a-b}{\sqrt{y_2^2 + (a-b)^2}}$$

If we consult again our original diagram, we see that we can also express this as

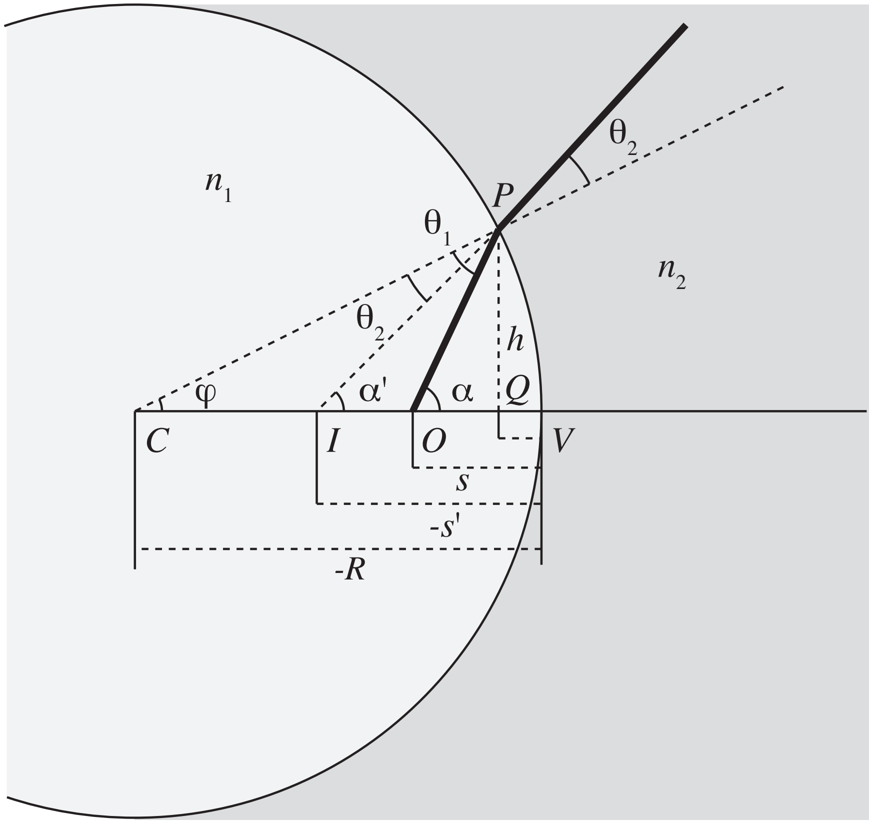
$$\begin{aligned} 0 &= n_1 \sin\theta_1 - n_2 \sin\theta_2 \\ \text{(4)} \quad \text{giving} \qquad n_1 \sin\theta_1 &= n_2 \sin\theta_2 \end{aligned}$$

so we have proved Snell's law.



# *Refraction at a curved interface*

Consider refraction at an interface with a radius of curvature $-R$. We'll follow a ray from the point $O$ as it travels at some angle $\alpha$ and hits the interface at a point $P$. Now Snell's law tells us



$$
\begin{aligned}
n_1 \sin\theta_1 &= n_2 \sin\theta_2 \\
n_1\theta_1 &\simeq n_2\theta_2 \\
n_1(\alpha - \phi) &\simeq n_2(\alpha' - \phi) \qquad (5)
\end{aligned}
$$

where we have used the small angle approximation. We get the relationship $\theta_1 = \alpha - \phi$ from considering the triangles $PCQ$, $CPO$, and $POQ$. We get the relationship $\theta_2 = \alpha' - \phi$ from considering the triangles $PCQ$, $CPI$, and $PIQ$.

# Refraction II

Again, we had from Eq. 5 the relationship

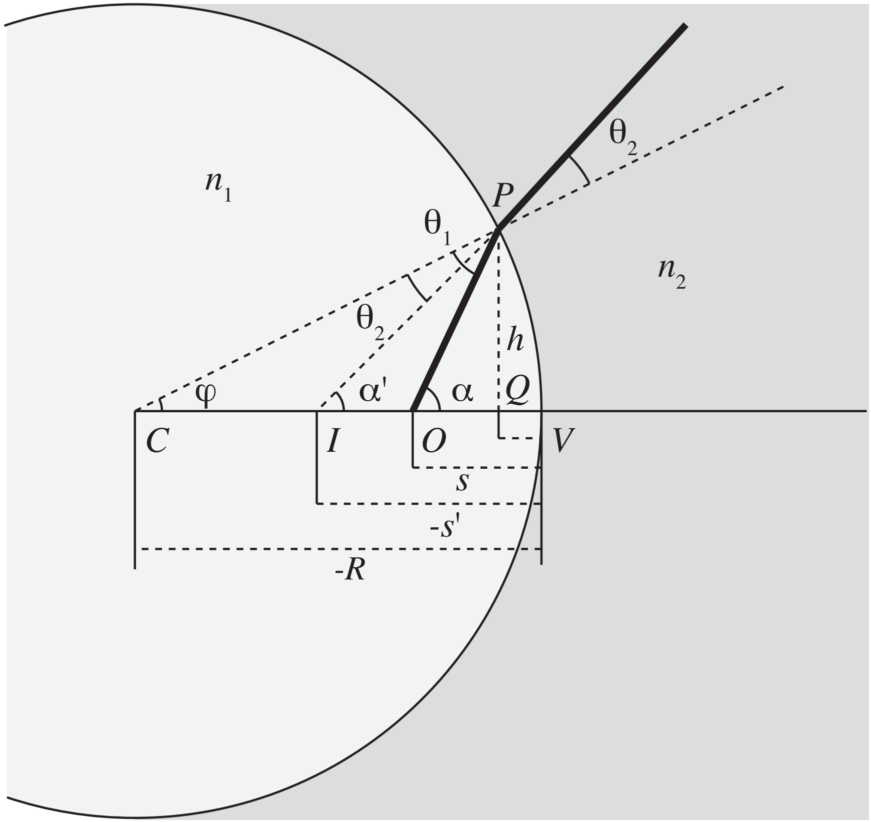
$$n_1(\alpha - \phi) = n_2(\alpha' - \phi)$$

in the small angle approximation. Now let's express these angles in terms of distances on the diagram, where we ignore the distance $QV$:

$$n_1\left(\frac{h}{s} - \frac{h}{-R}\right) \quad = \quad n_2\left(\frac{h}{-s'} - \frac{h}{-R}\right)$$

$$(6) \qquad \frac{n_1}{s} + \frac{n_2}{s'} \quad = \quad \frac{n_2 - n_1}{R} \text{ (Fowles 10.4)}$$

Notice that we defined the distances $s'$ and $R$ to be negative in anticipation of what will follow. This is a general relationship for refraction at a curved interface.



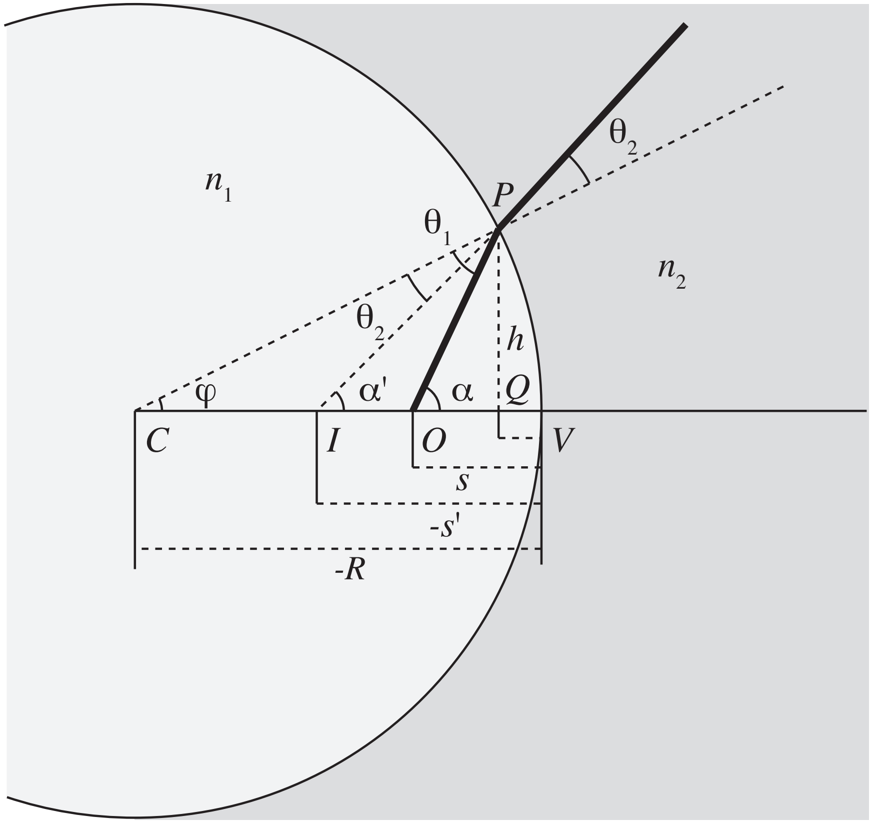
# *Apparent depth*

Let's consider some consequences of the expression of Eq. 6 of

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}.$$

First of all, consider standing on the side of $n_2$ and looking at an object at point $O$ within $n_1$, and letting $R \rightarrow \infty$ for a flat interface. As you look at the object and as your brain assumes that light travels in straight lines, you will think that the object is really located at point $I$. That is, the apparent depth $-s'$ of the object will be at

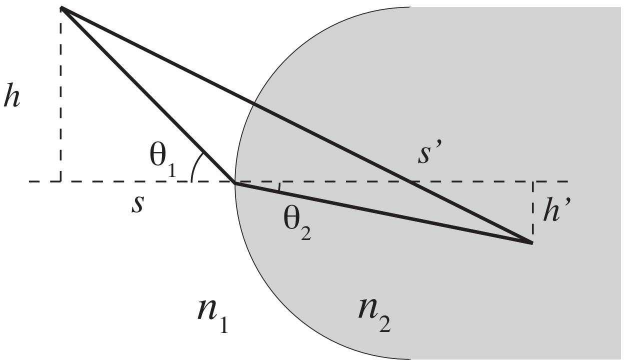
$$\begin{aligned} \frac{n_1}{s} &= -\frac{n_2}{s'} \\ s' &= -\frac{n_2}{n_1} s \end{aligned} \tag{7}$$



so if you're trying to spear a fish you will miss!

# *Magnification*

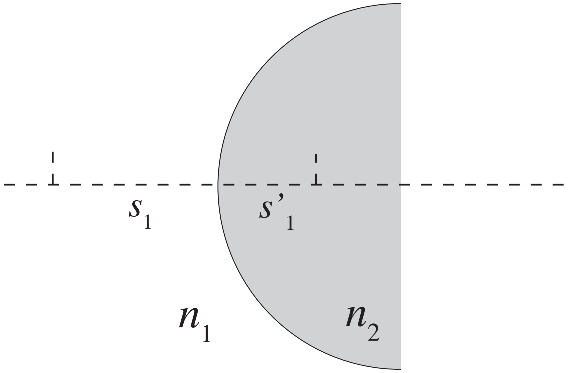
Now let's consider multiple rays from one point to another, hitting a convex refractive surface with positive $R$. One ray hits the spherical surface at $\theta_1 = 0$ so that $\theta_2 = 0$. Another ray goes through the centerline of the lens so that the angle $\theta_1$ from the source to the refractive interface is equal (in the small angle approximation) to $h/s$. The two rays cross at a point $s'$ on the other side of the surface (hence our choice of sign convention earlier for $s'$). The second ray reaches that point at an angle $\theta_2 = h'/s'$. From Snell's law we have



$$\begin{aligned} n_1 \frac{h}{s} &= n_2 \frac{h'}{s'} \\ \text{giving} \quad m \equiv \frac{h'}{h} &= -\frac{n_1 s'}{n_2 s} \end{aligned} \tag{8}$$

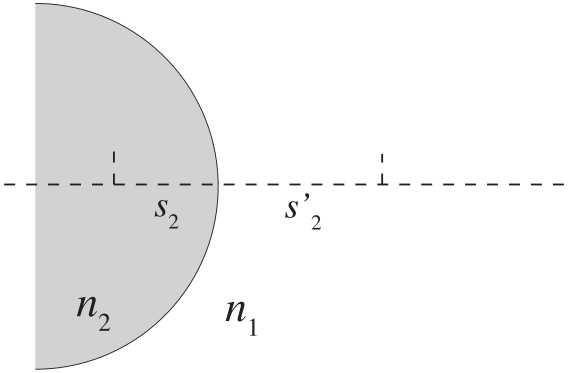
where the negative sign accounts for the fact that the object is inverted.

# *Two refractive surfaces*



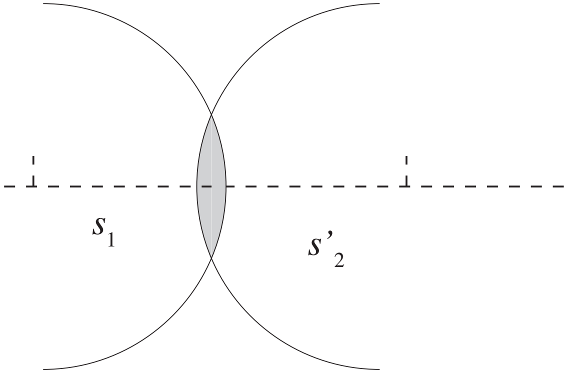
Consider two refractive surfaces. For the first surface we have

(9) $$\frac{n_1}{s_1} + \frac{n_2}{s'_1} = \frac{n_2 - n_1}{R_1}$$



Consider a second surface immediately following. For it, we have the opposite ordering for $n_1$ and $n_2$, and we have the opposite radius of curvature, giving

(10) $$\frac{n_2}{s_2} + \frac{n_1}{s'_2} = \frac{n_1 - n_2}{-R_2}$$



Here's the key: if the lens is truly thin, then

(11) $$s'_1 = -s_2.$$

# *Two surfaces II*

We can rewrite our expression of Eq. 9 for the first refractive interface as

$$\frac{1}{s_1'} = \frac{n_2 - n_1}{n_2}\frac{1}{R_1} - \frac{n_1}{n_2}\frac{1}{s_1} \tag{12}$$

and our expression of Eq. 10 for the second refractive interface as

$$\frac{1}{s_2} = \frac{n_1 - n_2}{n_2}\frac{1}{R_2} - \frac{n_1}{n_2}\frac{1}{s_2'} \tag{13}$$

Now let's use the result of Eq. 11 of $s_1' = -s_2$ to combine these two:

$$\begin{aligned}
\frac{n_1 - n_2}{n_2}\frac{1}{R_1} + \frac{n_1}{n_2}\frac{1}{s_1} &= \frac{n_1 - n_2}{n_2}\frac{1}{R_2} - \frac{n_1}{n_2}\frac{1}{s_2'} \\
\frac{n_1}{n_2}\left(\frac{1}{s_1} + \frac{1}{s_2'}\right) &= \frac{n_1 - n_2}{n_2}\left(-\frac{1}{R_1} + \frac{1}{R_2}\right) \\
\frac{1}{s_1} + \frac{1}{s_2'} &= \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \equiv \frac{1}{f} \qquad \text{(Fowles 10.7)}
\end{aligned}$$

# *Two surfaces III*

For the overall optical system, we'll write $s$ for $s_1$ and $s'$ for $s'_2$. Therefore we have derived the lensmaker's equation in Eq. 14:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \qquad \text{with} \qquad \frac{1}{f} \equiv \frac{n_2 - n_1}{n_1} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \equiv \frac{1}{f}$$

This allows us to calculate the focal length of a lens with any set of spherical surfaces. This equation assumes $+R$ for centers of curvature located “downstream” of the lens, so for a double-convex lens we have $R_1 = +|R_1|$ and $R_2 = -|R_2|$.