

# Restoring forces

- Consider an object that experiences a position-dependent force  $F(x)$ . For small displacements about some equilibrium position  $x_0$ , we can always consider a Taylor series expansion of the force:

$$F(x) = F(x_0) + (x - x_0) \frac{d}{dx} F(x)|_{x=x_0} + \frac{(x - x_0)^2}{2!} \frac{d^2}{dx^2} F(x)|_{x=x_0} + \frac{(x - x_0)^3}{3!} \frac{d^3}{dx^3} F(x)|_{x=x_0} + \dots$$

Let's take the case of  $x_0 = 0$  as the equilibrium position, and roll the terms  $(1/n!)(d^n/dx^n)F(x)|_{x=0}$  into coefficients  $k_n$ . We then have

$$F(x) \simeq k_0 + k_1x + k_2x^2 + k_3x^3 + \dots$$

If  $x_0 = 0$  represents an equilibrium position, then we have to have  $k_0 = 0$ . Also, if it's an equilibrium position, we have to have a negative value for  $k_1$ .

## Restoring forces II

- Again, we had

$$F(x) \simeq k_0 + k_1x + k_2x^2 + k_3x^3 + \dots$$

- What about  $k_2$ ? If it's a positive term, then we have a force pushing to the right for both leftward and rightward displacements from equilibrium, which does not sound like an equilibrium at all. Same story if it's negative. Therefore we must have  $k_2$  be zero or at least very small if we have an equilibrium (same for  $k_4$ ,  $k_6$ , and so on).
- We might have non-zero terms for  $k_3$ ,  $k_5$ , and so on. However, they must be either negative for a restoring force, or if positive they must be small compared to  $-k_1x$ . Because these terms involve higher derivatives of the force, and because they are divided by  $n!$ , we will assume they can be small.

# Simple restoring forces

- OK, we've determined that if a particle has an equilibrium position  $x_0 = 0$  then the force it experiences can be approximated as

$$F(x) \simeq -kx$$

at least for small displacements.

- Newton tells us that this is mass times acceleration:

$$(1) \quad m \frac{d^2 x}{dt^2} = -kx$$

A good trial to this differential equation is  $x = A \sin(\omega t + \varphi_0)$ . Let's try it out:

$$\begin{aligned} x &= A \sin(\omega t + \varphi_0) \\ x' = \frac{dx}{dt} &= \omega A \cos(\omega t + \varphi_0) \\ x'' = \frac{d^2 x}{dt^2} &= -\omega^2 A \sin(\omega t + \varphi_0) \end{aligned}$$

## Simple harmonic motion II

- Insert  $d^2x/dt^2 = -\omega^2 A \sin(\omega t + \varphi_0)$  into Eq. 1 of  $m(d^2x/dt^2) = -kx$  to obtain

$$-m\omega^2 A \sin(\omega t + \varphi_0) = -kA \sin(\omega t + \varphi_0)$$

from which we find

$$(2) \quad \begin{aligned} m\omega^2 &= k \\ \omega &= \sqrt{\frac{k}{m}} \end{aligned}$$

- We could have also chosen a cosine solution, but since  $\cos(x - \pi/2) = \sin(x)$  the only difference would be in the value of the starting phase  $\varphi_0$ .

# Harmonic oscillator

- We have found that a simple restoring force with equilibrium position  $x_0 = 0$  is well described by simple harmonic motion (referred to by French as SHM, but you should not confuse this with “single Hispanic male” in the personals section of the want ads):

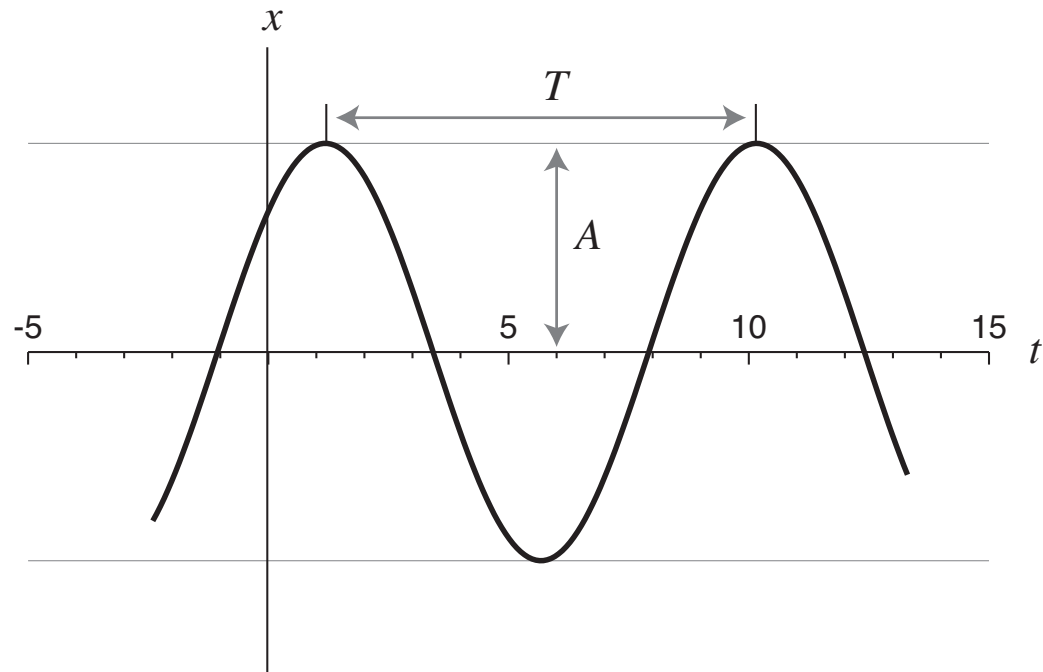
$$x = A \sin(\omega t + \varphi_0) \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}$$

- How to interpret the coefficient  $\omega$ ? Well, if we displace the particle it returns to the same displacement position when  $\omega t - \omega t_0 = 2\pi$ . This defines a period of oscillation  $T$  to be

$$(3) \quad T = \frac{2\pi}{\omega}$$

giving  $\omega = 2\pi/T$  as angular frequency (expressed in radians per second). We often prefer to talk about a frequency in cycles per second or Hertz, or  $f = \omega/2\pi = 1/T$ .

## Determining coefficients



- Besides determining the period, we can measure the amplitude  $A$  from measuring the maximum displacement  $x$  from  $x = 0$ . The total range of motion is from  $-A$  to  $+A$ .

## Determining coefficients II

- How to determine  $\varphi_0$ ? Well, if we know  $x_{t=0}$ , we can say

$$x_{t=0} = A \sin(\omega \cdot 0 + \varphi_0)$$

$$\frac{x_{t=0}}{A} = \sin(\varphi_0)$$

$$\varphi_0 = \sin^{-1}\left(\frac{x_{t=0}}{A}\right)$$

Alternatively we can find a time  $t_{x=0}$  where the position is zero:

$$0 = A \sin(\omega t_{x=0} + \varphi_0)$$

$$n\pi = \omega t_{x=0} + \varphi_0$$

$$\varphi_0 = n\pi - \omega t_{x=0}$$

If we also know the velocity  $dx/dt$ , we will want to use this as well in determining  $\varphi_0$  (the net effect will be to change to  $\varphi_0 = (2n + n_0)\pi - \omega t_{x=0}$  where  $n$  is an integer and  $n_0$  is 0 or 1).

# Complex algebra for wave equations

- Our equation for periodic motion is  $x = A \sin(\omega t + \varphi_0)$ . This is not an entirely satisfactory way of writing things.

Consider the case when  $\omega t + \varphi_0 = 2n\pi$  such that  $x \rightarrow 0$ .

- A good way to do this is to use complex notation:

$$x = \operatorname{Re} \left[ A \exp(i\omega t + \varphi_0) \right] = \operatorname{Re} [A e^{i\omega t + \varphi_0}]$$

- Phase lets us keep track of whether we're at the max, zero, or min of the wavefield.

[Look at the movie.](#)

- To be continued. . .

