PHY 300, Spring 2006, Exam 3, May 15, 2006
There are seven problems in total! Please show all work in your exam book. Calculators are allowed. You have been given an equation sheet. Do the easiest problem first, and the hardest last.

1. A telescope with 5 m aperture and 10 m focal length is used to view the moon on a clear, cold night so that atmospheric turbulence can be neglected. What's the finest detail the telescope can resolve on the moon, which is 385,000 km away?

Answer: image resolution is determined by the numerical aperture which is N.A.=$n\sin\theta$ where θ is the opening angle of the lens relative to the optical axis (that is, it's the semi-angle rather than the full angle which is double the semi-angle). The semi-angle of the telescope is $\theta = \arctan(2.5/10) = 14°$, so its numerical aperture is N.A.=$1 \cdot \sin(14°) = 0.243$. With $\lambda = 500$ nm light, its resolution at its own image plane is is $0.61\lambda/\text{N.A.} = 0.61 \cdot 500 \times 10^{-9}/0.243 = 1.26\ \mu$m. We can relate this image height h' to an object height h with

$$\frac{h'}{h} = -m = \frac{s'}{s} \qquad \Rightarrow \qquad h = h'\frac{s}{s'} = (1.26 \times 10^{-6})\frac{3.85 \times 10^8}{10} = 48.3 \text{ m}$$

so we can't see any detail smaller than about half a football field with this telescope, even on a clear night.

2. When an object of mass 0.2 kg is hung from a spring, the spring stretches by 2.45 cm. When the object is pulled down and released, it is observed that it has a velocity-dependant damping force $-bv$.
   (a) What's the object's differential equation of motion?
   (b) If the damped frequency is $\sqrt{3}/2$ of the undamped frequency, what is the value of the constant b?
   (c) What is the Q of the system?

Answer: this is pretty much like French problem 3.14.

(a) The differential equation is

$$mx'' + bx' + kx = 0$$

with $m = 0.2$ kg, and $k = F/x = mg/x = 0.2 \cdot 9.8/0.0245 = 80$ N/m.

(b) From French Eq. 3.34 we find that the damped motion is at a frequency ω which differs from the undamped resonance frequency $\omega_0 = \sqrt{k/m}$ according to

$$\begin{aligned}
\omega^2 &= \omega_0^2 - \frac{b^2}{4m^2} \\
\frac{b^2}{4m^2} &= \omega_0^2 - \omega^2 \\
b &= 2m\sqrt{(\omega_0^2 - \omega^2)} = 2m\omega_0\Big(1 - (\frac{\omega}{\omega_0})^2\Big)^{1/2} \\
&= 2m\sqrt{\frac{k}{m}}\Big(1 - (\frac{\omega}{\omega_0})^2\Big)^{1/2} = 2\sqrt{km}\Big(1 - (\frac{\omega}{\omega_0})^2\Big)^{1/2}
\end{aligned}$$

Now $\omega = (\sqrt{3}/2)\omega_0$ so we have

$$b = 2\sqrt{80 \cdot 0.2}\Big(1 - (\frac{\sqrt{3}}{2})^2\Big)^{1/2} = \sqrt{80 \cdot 0.2} = 4 \text{ Ns/m}$$

(c) We can get Q from

$$Q = \frac{\omega_0}{\gamma} = \frac{m\omega_0}{b} = \frac{\sqrt{km}}{b} = \frac{\sqrt{80 \cdot 0.2}}{4} = 1$$

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3. A coupled oscillator system consists of two 0.1 kg balls on the end of lightweight rods of length 0.3 m, connected by a spring with constant $k = 1$ N/m. Describe the modes of motion of the system, and give quantitative results for their frequencies in Hz.

Answer: This is like French 5.2 but simpler. For each individual pendulum, the resonance frequency is $\omega_0 = \sqrt{g/\ell} = \sqrt{9.8/0.3} = 5.72$ radians/sec or $f = 5.72/(2\pi) = 0.91$ Hz. The coupling involves a frequency $\omega_c = \sqrt{k/m} = \sqrt{1/0.1} = 3.17$ radians/sec or $f = 3.17/(2\pi) = 0.505$ Hz. The net motion of the system is to either go in common mode (both masses moving together) at 0.91 Hz, or in differential mode (the two masses oscillating in opposite directions) at an angular frequency of

$$\omega' = \sqrt{\omega_0^2 + 2\omega_c^2} = \sqrt{5.72^2 + 2 \cdot 3.17^2} = 7.26 \text{ rad/sec}$$

or $f' = 1.16$ Hz.

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4. A double-slit diffraction pattern is formed using HeNe laser light at 632 nm. Each slit has a width of 0.080 mm. The pattern reveals that the third-order interference maxima are missing from the pattern. What is the slit separation? What is the irradiance of the first two interference fringes, relative to the zero$^{\text{th}}$-order maximum?

Answer: For double-slit diffraction we have $I \propto (\sin^2\beta/\beta^2)\cos^2\alpha$ with $\beta = \pi b f$ and $\alpha = \pi a f$. We also know that the slit width is $b = 0.080$ mm, and $\lambda = 632$ nm. Interference maxima occur when $\alpha = m\pi$ for the m^{th} order, and we're told that the 3^{rd} interference maxima are missing. We therefore must have a diffraction minimum coinciding with the interference maximum. Diffraction minima are at $\beta = n\pi$. Our conditions are

$$\beta = n\pi = \pi b f \text{ or } f = \frac{n}{b} \qquad \text{and} \qquad \alpha = m\pi = \pi a f \text{ or } f = \frac{m}{a}$$

and we must have the two spatial frequencies f conincide, so we have $(n/b) = (m/a)$ or $a = (m/n)b = (3/n)b$. Assuming $n = 1$ we have $a = (3/1)b = 0.240$ mm. The irradiance of the first two fringes can be found from knowing that

$$\alpha_{1,2} = \{1, 2\}\pi$$

and

$$\beta = \pi b f = \pi b \frac{m}{a} = \pi\left(\frac{b}{a}\right)m = \pi\left(\frac{0.080 \text{ mm}}{0.240 \text{ mm}}\right)m = \pi\frac{m}{3} = \pi\frac{\{1, 2\}}{3}$$

which leads to

$$\left(\frac{\sin(\pi\{1,2\}/3)}{\pi\{1,2\}/3}\right)^2 \cos^2(\{1,2\}\pi) = 0.684, 0.171$$

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5. You want to distinguish between $\lambda = 500.2$ and 500.3 nm light with a grating at normal incidence to the light and with a maximum angle for third order diffraction of $5°$. What is the minimum width of the grating?

Answer: To get third-order diffraction at $5°$, the grating period d should satisfy $d \sin\theta = 3\lambda$ or

$$d = \frac{3\lambda}{\sin\theta} = \frac{3 \cdot 500.25 \times 10^{-9}}{\sin 5°} = 1.72 \times 10^{-5} \text{ m}$$

or 17.2 μm. To distinguish 0.1 parts out of 500.25 mean wavelength, we need at least 5000 waves of path length difference or 5000/3 grating bars, so the minimum width is

$$\frac{5000}{3} \cdot 1.72 \times 10^{-5} = 2.87 \times 10^{-2} \text{ m}$$

or 2.87 cm.

Note that several people used $m\lambda = a\sin(2\theta_b)$ to solve this, but this is an expression for calculating the blaze angle θ_b of a reflection grating rather than the period a of the grating based on the diffraction angle θ. Wrong formula for this application!

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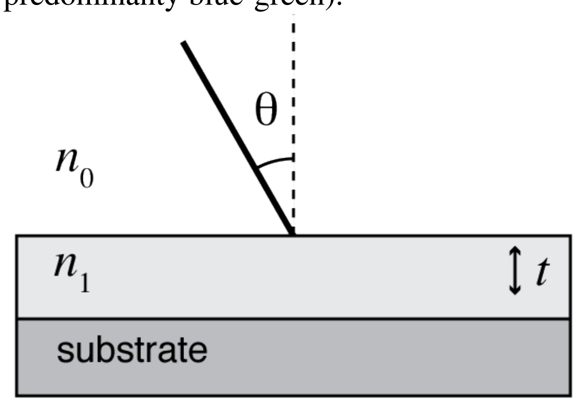
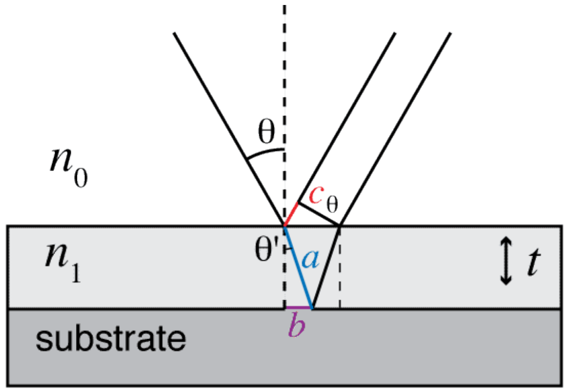
6. Parallel light is incident on a dielectric film on a nontransparent substrate such that there is relatively weak reflectivity from each interface, and that there are no sign changes upon reflection. For visible light of wavelength λ incident at an angle θ, what thickness t of the layer n_1 gives rise to a minimum in reflectivity? Assume $n_0 = 1.00$. Derive a general formula, and give a numerical result for t for $\theta = 30°$, $n_1 = 1.2$, and a dip in reflectivity at $\lambda = 600$ nm (making the film look predominanty blue-green).



Answer: because the reflectivity from each layer is relatively weak, we will not have to worry about multiple bounces inside the thin film so we can solve the problem in terms of two different single reflectivities:



The path length difference between the two waves will be $\Delta = 2n_1 a - c$, so we want to find the lengths a and c. The length of c is given by

$$c = 2b\sin\theta \qquad \text{so} \qquad \Delta = 2n_1 a - c = 2n_1 a - 2b\sin\theta.$$

We can also find relationships between a, b, and t from trigonometry:

$$b = a\sin\theta' \qquad \text{and} \qquad \frac{b}{t} = \tan\theta'.$$

Finally, Snell's law tells us

$$n_0\sin\theta = n_1\sin\theta' \qquad \text{or} \qquad \sin\theta' = \frac{\sin\theta}{n_1}$$

since $n_0 = 1$. Again, what we want to find in the end is an expression for the path length difference Δ:

$$\begin{aligned}
\Delta &= 2n_1 a - 2b\sin\theta = 2n_1\frac{b}{\sin\theta'} - 2b\sin\theta = 2n_1^2\frac{b}{\sin\theta} - 2b\sin\theta \\
&= 2b\left(\frac{n_1^2}{\sin\theta} - \sin\theta\right) = 2t\tan\theta'\left(\frac{n_1^2}{\sin\theta} - \sin\theta\right) = 2t\frac{\sin\theta'}{\sqrt{1-\sin^2\theta'}}\left(\frac{n_1^2}{\sin\theta} - \sin\theta\right) \\
&= 2t\frac{\sin\theta}{n_1}\frac{1}{\sqrt{1-\sin^2\theta/n_1^2}}\left(\frac{n_1^2}{\sin\theta} - \sin\theta\right) = 2t\frac{\sin\theta}{n_1}\frac{n_1}{\sqrt{n_1^2-\sin^2\theta}}\left(\frac{n_1^2}{\sin\theta} - \sin\theta\right) \\
&= 2t\frac{1}{\sqrt{n_1^2-\sin^2\theta}}\left(n_1^2 - \sin^2\theta\right) = 2t\sqrt{n_1^2-\sin^2\theta}.
\end{aligned}$$

Now that we know the optical path length difference, we can set the condition for destructive interference as $2\pi\Delta/\lambda = \pi$, giving

$$\Delta = 2t\sqrt{n_1^2-\sin^2\theta} = \frac{\lambda}{2} \qquad \text{or} \qquad t = \frac{\lambda}{4\sqrt{n_1^2-\sin^2\theta}}$$

With $\lambda = 600$ nm, $\theta = 30^\circ$, and $n_1 = 1.2$, we have

$$t = \frac{600\text{ nm}}{4\sqrt{1.2^2-\sin^2 30^\circ}} = 137\text{ nm}$$

as the thickness of a film that produces a minimum in reflectivity at 600 nm and thus has a blue-green tinge to its color when illuminated with white light.

Note that some people wanted to use the formula

$$\delta = 4\pi \frac{\ell}{\lambda} \frac{n_t}{\sqrt{1-(n/n_t)^2 \sin^2 \theta_i}}$$

for phase of the reflected light from multiple reflections in a glass slab, without knowing what the formula meant. For this problem, you had to think and reason it out without using a canned formula!!!

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7. Describe how the refractive index comes about, what mechanical model it can be described in terms of, what sets the low and high frequency limit, and the form of the refractive index at low frequencies. Your answer can include figures, diagrams, etc., and it should fit within three pages in your blue book (when the blue book is open, you have two pages open in front of you).